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UNIVERSITAT POLITÈCNICA DE CATALUNYA
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UPCGRAU

Fundamental Mechanics.
Newtonian Mechanics for Engineering

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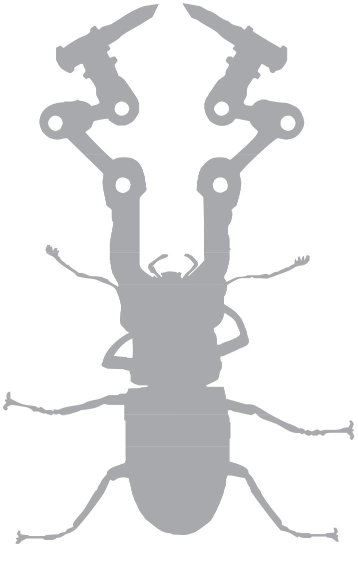
Contents



Prologue

Mechanics is a branch of physics that deals with the relationships that physical objects have with space and time. In dealing with such basic notions, we must consider the specific characteristics and problems of mechanics:

- Mechanics acts as the metalanguage[1] of physics. Mechanics is where we must define what we mean by matter, space and time. Although concepts such as force and energy are used in many branches of physics, it is mechanics that provides the first definition upon which the others are based.

[1] A metalanguage is an artificial language formed by a repertoire of axioms, postulates and rules, that are used to describe other languages, whether artificial or natural.

- Because mechanics deals with very basic concepts, they are the same concepts used in everyday life, detached from physics. So, words such as *length, distance, time, speed, force, work, energy* and the selfsame *mechanics* are the same words we use daily in very different contexts. When operating in everyday contexts, we do not always function rationally, and even less so scientifically, which is highly recommended. The downside to this is that we generate a whole series of intuitions that we unwittingly use in the scientific context of mechanics, which is not recommended.

- The concept of mechanics is also used in many different contexts within and beyond physics. Thus, we have:

  – **Classical mechanics**. This deals with objects whose dimensions are much larger than atoms and molecules. In this field, we can take spacetime measurements without the problem of substantially altering them.

  – **Quantum mechanics**. This serves as the complementary to the classical mechanics and deals with extremely small objects, that are on the order of atoms and molecules. These objects pose problems when taking spatio-temporal measurements because doing so alters them substantially. We do not know how to measure these objects because, according to quantum mechanics, it is not possible to do so without altering them.

- **Newtonian mechanics**. This is the field of classical mechanics that deals with objects moving at low speeds, compared to that of light.
- **Relativistic mechanics**. This field deals with objects moving at high speeds comparable to that of light. It can be classical, like Newtonian, or quantum. Even today, attempts are still being made to establish the foundations of relativistic quantum mechanics.
- **Other contexts.** Some of these are continuum mechanics, fluid mechanics, rigid body mechanics, statistical mechanics, analytical mechanics, field mechanics, mechanical engineering, biomechanics and many others.

From all of these, we will deal here with Newtonian mechanics, which is a non-relativistic classical mechanics approach to dealing with objects that are neither extremely small nor excessively fast. This embraces a good part of the world around us. To understand it better, let us describe Newtonian mechanics as the *mechanics of the everyday world*. However, we must be very careful to not fall into unsubstantiated intuitions.

Chapter 1 deals with the most basic foundation of mechanics. It concerns what we understand by space and time, as well as which forms of mathematics will be useful for describing and measuring the magnitudes of mechanics. We will characterize the curves drawn by points in space and their relationships to time, specifically by studying kinematics, which is an extension of the usual space geometry to include time. This chapter will inroduce concepts such as reference frame, Newton's first law, position, velocity and acceleration, among others.

Chapter 2 seeks to define the key object in Newtonian mechanics: the *Newtonian* particle. This concept will later allow us to construct objects that are closer to real ones, such as rigid bodies. This is the objective for which we define the Newtonian particle. We will deal with particle dynamics, meaning the time trajectory traced by the particle when subjected to different forces. This chapter introduces Newton's law of motion and concepts such as momentum, angular momentum, force, moment of a force, work and energy, among others.

Chapter 3 can be summarized as *getting to the point*. We extend the concepts introduced in the previous chapter to a set of N particles. Regarding N, it can be understood as a finite number $(1, 2, 3...)$ or can even be infinite. In the latter case, we will be dealing with continuous bodies and will see how they can be handled without problems by using the concepts introduced for N finite. The chapter will begin by distinguishing between internal and external forces, before looking at Newton's law of action-reaction. We will introduce the fundamental concept of constraints and ideal reactions, thus bringing us to the core component of mechanics: the *general equation of dynamics*. This equation is important because it allows handling

complex mechanical systems, which engineers call *mechanisms* or *machines*. With this equation, we can make the most of our knowledge of the mechanism's geometry and eliminate the reaction forces generated by this mechanism. In conservative systems with one degree of freedom, we can use the properties of the *mechanical energy function for the system* as a solving method. For systems with more than one degree of freedom and/or with time-dependent constraints (which are beyond the scope of this book), the general equation of dynamics would lead us to Lagrangian mechanics, which we introduce in Chapter 9 for those readers eager to learn about mechanics. Chapter 3 will also introduce the *rigid body*, which is a system of constrained particles. We do not deal here with the dynamics of the rigid body but instead will postpone this to Chapter 5, while Chapter 4 will deal with the statics of rigid bodies.
Chapter 3 will also introduce the general concept of *angular* velocity.

As already mentioned, Chapter 4 deals with statics or, stated more generally, with equilibrium problems of rigid body systems, which includes studying the stability of this equilibrium. We also deal with how to correctly represent some external forces acting on bodies, such as weight and forces exerted by gravitating fluids in contact with bodies. The statics problem is posed as an inverse problem of mechanics. If, in general, the standard problem in mechanics is to find the motion of a system subjected to known forces, this *motion* is now known, since we want it to remain at rest. Beyond the reaction forces, the unknowns will be some other forces or also the configuration of the system itself.

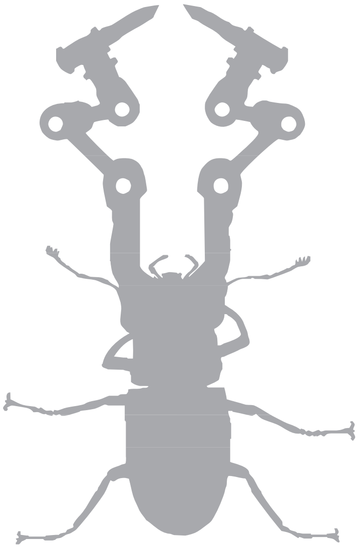
Chapter 5 deals with the dynamics of the rigid body. For this, it will be necessary to finish expressing the equations of motion found in Chapter 3 as a function of the appropriate degrees of freedom. Due to the difficulty, we will restrict our study to situations where the axis of rotation is in a fixed direction of space, which we will call *2 D rotations*.
Here is where we introduce the concept of *moment of inertia about an axis*.

Chapter 6 deals with the problem of small oscillations in systems with one degree of freedom. We will see how this problem is of great interest, because any system whose equilibrium is stable for small deviations from this configuration performs a harmonic motion whose characteristics are independent of any exact knowledge we have of the acting forces. For this reason, we will apply a certain degree of abstraction to the study and introduce the concept of the canonical equation of harmonic motion, specifically for simple, damped and forced cases. This chapter also introduces concepts such as elongation, pulsation, period, amplitude and phase, as well as the important concept of resonance.

Chapters 7 and 8 are dedicated to the study of wave phenomena, which are understood as mechanical phenomena. The study of wave motion is an extension of Chapter 6's study of harmonic motion, but because it is complex and encompasses a variety of perceptible phenomena, we treat it in a way that is closer to phenomenology than to pure formal mathematical development.

As already mentioned, Chapter 9 serves only as a taste for anybody interested in analytical mechanics and how it relates to Newtonian mechanics through the general equation of dynamics. Beginning with this, we deduce the Lagrange equations of motion for systems with time-dependent geometric constraints, which are an extension to L degrees of freedom of Chapter 3's equations for conservative systems using the mechanical energy function, which in a certain way will now be replaced by the Lagrange or Lagrangian function. We do not delve deeper into this because our only objective is to create a bridge between Newtonian and Lagrangian mechanics, which is a part of analytical mechanics that can be found in a multitude of texts.

Finally, we point out that each chapter contains some solved problems that help to contextualize the developed theories. The final part of the book provides more problems and questions arranged by chapters and sections. Some of the problems are solved, and the solutions are given for the rest.



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1 Physico-mathematical foundations of mechanics

Introduction

This chapter combines physics and mathematics because it is a very fine line that separates the most basic concepts of physics from mathematics.

1.1 Space, time and reference frames

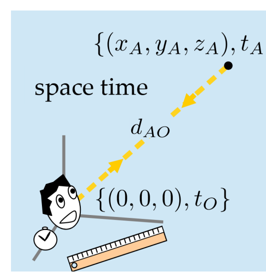


Fig. 1.1: Observer

Physics in general, and mechanics in particular, study the behaviour of objects in space time. Although some current physical theories debate the reality of this spacetime, we will not do so here. It is enough to view spacetime as a very useful intellectual construction for systematising everything that happens, as we all agree on how to organize our perceptions of events by placing them at different instants of time in a three-dimensional space.

An **observer** (see Figure 1.1) is a recording system (human or automatic) of the position of objects at each **instant** in space. He or she can assign a triad of numbers to each point A which will apply a unique label to it: the coordinates of point $A = (x_A, y_A, z_A)$. He or she will be able to assign to each point A a time t_A, which is **synchronized** with the time of clock t_O using the **maximum available speed** c:

$$t_A - t_O = d_{AO}/c \tag{1.1}$$

where d_{AO} is the **distance** between A and the point where the observer's clock is located. Later, we will see that using any speed other than the maximum gives rise to inconsistencies.

The set of objects and labels used by the observer constitute the **reference frame**.

By international agreement, all observers define time and space patterns in the same way. The BIPM (Bureau International des Poids et Mesures) is the institution that oversees the SI (International System of Units). The final definitions are:

Unit of time. The second, s: a time interval of 1 second is equal to 9192631770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom (Adopted in 1967).

Unit of distance. The meter, m: a spatial interval of 1 meter is equal to the distance travelled by light in vacuum during a time interval of 1/299792458 s (Adopted in 1983).

1.2 Euclidean Cartesian coordinates. Distance

A remarkable fact about physical space is that it can be labelled using **Euclidean Cartesian coordinates** (simply Cartesian, if there is no doubt). An important property of these coordinates is that the distance, or shortest path between two points, $A = (x_A, y_A, z_A)$ and $B = (x_B, y_B, z_B)$ (see Figures 1.2 and 1.3) can be expressed as

$$d(A,B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2} \tag{1.2}$$

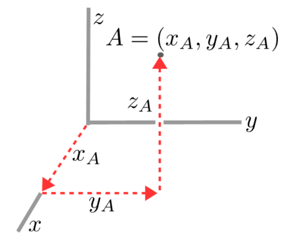


Fig. 1.2: Euclidean Cartesian coordinates of the A point

This is not possible for the space points on the surface of a sphere.

Point $(0, 0, 0)$ is the **origin of coordinates**; and the straight lines $(x, 0, 0)$, $(0, y, 0)$ and $(0, 0, z)$ are the **coordinate axes**.

1.3 Scalars and vectors

Physical quantities have some characteristics in relation to the observer measuring them. According to this criterion, we highlight and use two categories of physical magnitudes: scalars and vectors.

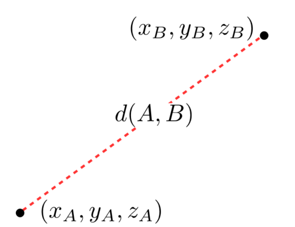


Fig. 1.3: Distance from point A to B

Scalar field is a map that associates points and instants with values that are independent of the observer's orientation (or rotation), $f(x, y, z, t)$. It is also called a *scalar*.

Vector field is a map that associates points and instants with vectors: a positive scalar (modulus or magnitude) and a direction (line of action and sign), $\vec{V}(x, y, z, t)$. It is also called a *vector*.

Examples of scalar magnitudes are temperature, volume and density, among others. Examples of vector magnitudes are the velocity of a particle associated with the points and instant it passes through, the gravitational field associated with all points at each instant, and the force exerted by an electrical charge wherever one may be.

The idea that a vector exists at each point in space is why we use the name of **field**. Let us imagine a wheat field in which each ear of wheat is the vector of a point in space. The set of ears of wheat, that is, the entire field of wheat, is what we call the ***vector field*** or simply *vector*. We can also speak of a **scalar field** such us a field of densities or temperatures, although not in the sense of a point's density or temperature in spacetime, but of a function that assigns at each point and at each instant a density or temperature that may be different.

The **Cartesian basis** of vectors is formed by the unit vectors $\left\{\hat{\imath}, \hat{\jmath}, \hat{k}\right\}$ that run parallel to the coordinate axes and are defined at all points in space (see Figure 1.4). Any vector $\vec{V}$ can be written as

$$\vec{V} = V_x\hat{\imath} + V_y\hat{\jmath} + V_z\hat{k} = (V_x, V_y, V_z) \tag{1.3}$$

where V_x , V_y and V_z, are the Cartesian components of the vector, which can be functions of (x, y, z, t).[1]

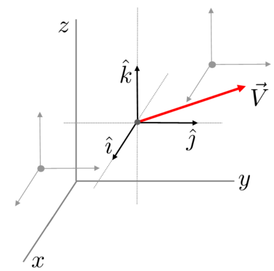


Fig. 1.4: At each point in space we have the three basis vectors $\left\{\hat{\imath}, \hat{\jmath}, \hat{k}\right\}$

[1] If any of the components is numeric and we are using a comma as a separator, we can use a semicolon (;) instead of a comma (,). For example, $(2; 5, 603; 7.2)$. In this book we will always use a comma

The basis vectors $\left\{\hat{\imath}, \hat{\jmath}, \hat{k}\right\}$ have no units and are not associated with any magnitude. They represent directions of space, which in this case are the three directions of the coordinate axes.

Note that a component of a vector is not a scalar.

Algebraic operations with vectors

If we have two vectors $\vec{A}$ and $\vec{B}$, it can be shown that $(A_x + B_x, A_y + B_y, A_z + B_z)$ is a vector. The sum of two vectors is defined according to this property.

➔ **Sum:** Given two vectors $\vec{A}$ and $\vec{B}$, their sum $\vec{A} + \vec{B}$ is the vector which, in Cartesian coordinates, is expressed as

$$\vec{A} + \vec{B} = (A_x + B_x, A_y + B_y, A_z + B_z) \tag{1.4}$$

If U is a scalar and $\vec{A}$ is a vector, it can be shown that (UA_x, UA_y, UA_z) is a vector. According to this, the external product or the product of a scalar and a vector can be defined.

➔ **External product:** Given a scalar U and a vector $\vec{A}$, the external product is the vector $U\vec{A}$, which in Cartesian coordinates can be expressed as

$$U\vec{A} = (UA_x, UA_y, UA_z) \tag{1.5}$$

If we have two vectors $\vec{A}$ and $\vec{B}$, it can be shown that $A_xB_x + A_yB_y + A_zB_z$ is a scalar. According to this, we define the scalar (or dot) product.

➔ **Scalar product:** Given two vectors $\vec{A}$ and $\vec{B}$, the scalar product is the scalar $\vec{A} \cdot \vec{B}$ which in Cartesian coordinates can be expressed as

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z \tag{1.6}$$

➔ The **modulus** of a vector $\vec{A}$ is the scalar:

$$A = \sqrt{\vec{A} \cdot \vec{A}} \tag{1.7}$$

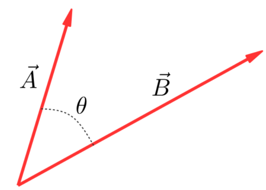


Fig. 1.5: Angle θ between two vectors

➔ The cosine of the angle $0 \leq \theta \leq \pi$ (see Figure 1.5) between two vectors $\vec{A}$ and $\vec{B}$ can be expressed as

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{AB} \tag{1.8}$$

➔ The **unit vector** $\hat{A}$ of a vector $\vec{A}$ is the unit module vector

$$\hat{A} = \frac{\vec{A}}{A} \tag{1.9}$$

Given two vectors $\vec{A}$ and $\vec{B}$, it can be proved that

$$(A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x)$$

is a vector. Accordingly, we have defined the vector product.

➔ **Vector product**: Given two vectors $\vec{A}$ and $\vec{B}$, the vector product is the vector $\vec{A} \times \vec{B}$ (see Figure 1.6), which in Cartesian coordinates can be expressed as

$$\vec{A} \times \vec{B} \equiv \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x) \tag{1.10}$$

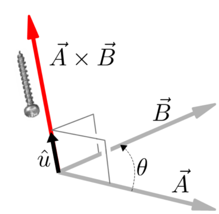


Fig. 1.6: Vector product $\vec{A} \times \vec{B}$ between two vectors. Corkscrew rule.

➔ The sine of the angle $0 \leq \theta \leq \pi$ between two vectors $\vec{A}$ and $\vec{B}$ can be expressed as

$$\sin\theta = \frac{\left|\vec{A} \times \vec{B}\right|}{AB} \tag{1.11}$$

➔ **Corkscrew rule**:The vector product $\vec{A} \times \vec{B}$ can be expressed as

$$\vec{A} \times \vec{B} = B\, A \sin\theta\, \hat{u} \tag{1.12}$$

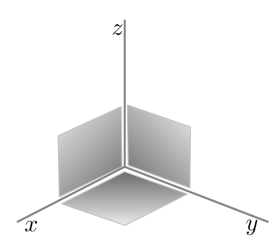


Fig. 1.7: Reference frame $\{x, y, z\}$ of positive orientation

where $\hat{u}$ is a unit vector normal to $\vec{A}$ and $\vec{B}$ the direction given by an advancing dextro-rotatory screw rotating from $\vec{A}$ to $\vec{B}$ on the shortest path (see Figure 1.6).

It is especially important to use $\{x, y, z\}$ reference frames of **positive orientation**, that is, frames that can be juxtaposed onto the frame in Figure 1.7 only by translation and rotation.

Algebraic relations with vectors

These are equalities (which we will give without proof) involving the above defined operations:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C} \tag{1.13}$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \tag{1.14}$$

$$(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C}) \tag{1.15}$$

Differential operations with vectors

Given a function $f(x, y, z, t)$, we may want to derive it with respect to some of the variables. For example, we will indicate its x derivative with the operator $\frac{\partial f}{\partial x}$ and will say that we make the **partial derivative with respect to** x. If we want to derive it only with respect to t, we will indicate it with $\frac{\partial f}{\partial t}$ and will say that we make the partial derivative with respect to t.

It must be remembered that the **chain rule** for functions states that if $F(\lambda) = f(g(\lambda), \lambda)$, then $\frac{d}{d\lambda}F = \frac{\partial F}{\partial g}\frac{\partial g}{\partial \lambda} + \frac{\partial F}{\partial \lambda}$.

When we derive vectors, we can use the usual rules of derivation, especially Leibniz's rule for the product. In general, scalars and vector components are functions of the type $f(x, y, z, t)$. In addition, it can happen that the coordinates $\{x, y, z\}$ are part of a trajectory and, therefore, may depend on t or on some other λ parameter: $\{x(t), y(t), z(t)\}$ or $\{x(\lambda), y(\lambda), z(\lambda)\}$. The derivation operator D may be: $D = \frac{\partial}{\partial x}$, $D = \frac{\partial}{\partial y}$, $D = \frac{\partial}{\partial z}$, $D = \frac{\partial}{\partial t}$, $D = \frac{d}{dt}$, $D = \frac{\partial}{\partial \lambda}$, $D = \frac{d}{d\lambda}$. All of them will comply with the usual rules of derivation and especially with the Leibniz rule.

The **Leibniz rule** for functions f and g with the usual product is $D(fg) = (Df)g + f(Dg)$. When the expressions involve scalars U and vectors $\vec{A}$ and $\vec{B}$ with the scalar $\cdot$ and vector $\times$ products, we have:

$$D(U\vec{A}) = (DU)\vec{A} + U(D\vec{A}) \tag{1.16}$$

$$D(\vec{A}\cdot\vec{B}) = (D\vec{A})\cdot\vec{B} + \vec{A}\cdot(D\vec{B}) \tag{1.17}$$

$$D(\vec{A}\times\vec{B}) = (D\vec{A})\times\vec{B} + \vec{A}\times(D\vec{B}) \tag{1.18}$$

An important property of the Cartesian basis (associated with the $\{x, y, z\}$ coordinates) $\left\{\hat{\imath}, \hat{\jmath}, \hat{k}\right\}$ is that

$$D\hat{\imath} = D\hat{\jmath} = D\hat{k} = 0 \tag{1.19}$$

This is so because at each space point the vectors of the fields $\left\{\hat{\imath}, \hat{\jmath}, \hat{k}\right\}$ remain parallel (see Figure 1.8).This does not happen if, for example, a spherical basis is used. At each point in space, the radial unit field vectors $\hat{r}$ do not remain parallel. Nor do the other two fields completing the spherical basis (see Figure 1.9).

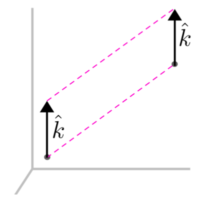


Fig. 1.8: The $\hat{k}$ basis vectors are parallel. So are $\hat{\imath}$ and $\hat{\jmath}$

Property (1.19) is very useful. By deriving a vector on a Cartesian basis

$$D\vec{V} = D(V_x\hat{\imath}) + D(V_y\hat{\jmath}) + D(V_z\hat{k}) = D(V_x)\hat{\imath} + D(V_y)\hat{\jmath} + D(V_z)\hat{k}$$

it is enough to derive its components:

$$D\vec{V} = (DV_x, DV_y, DV_z)$$

In the case of derivation with respect to t, it is interesting to note that the variables x, y and z may also depend on t because they are part of a trajectory. In other words, they are in fact functions of t, and the function f is a function only of t, $f(x(t), y(t), z(t), t)$, although we do not make it explicit. In this case, the derivative with respect to t is called the ***total derivative*** and is written as $\dot{f} = \frac{df}{dt}$. To calculate this, we will use the chain rule for each coordinate as well as the time partial derivative, i.e.:

$$\dot{f} = \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial z}\dot{z} + \frac{\partial f}{\partial t} \tag{1.20}$$

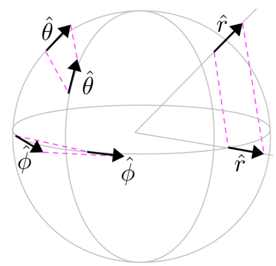


Fig. 1.9: The $\hat{r}$ basis vectors are not parallel. Nor are $\hat{\theta}$ and $\hat{\phi}$

Concerning total derivatives, we have the following two important results.

- Given a scalar f, the total derivative $\frac{df}{dt}$ is a scalar.
- Given a vector $\vec{v}$, the total derivative $\frac{d\vec{v}}{dt}$ is a vector.

Other basic differential operations between scalars and vectors are the following.

- Given a scalar U, the **gradient** $\vec{\nabla}U$ is the vector that in Cartesian coordinates can be written as

$$\vec{\nabla}U = \frac{\partial U}{\partial \vec{r}} \equiv \left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) \tag{1.21}$$

- The **differential** of $U(x, y, z)$ is the infinitesimal scalar expression dU, which in Cartesian coordinates can be written as

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz \tag{1.22}$$

We can also express as $dU = \vec{\nabla}U \cdot d\vec{r} = \frac{\partial U}{\partial \vec{r}} \cdot d\vec{r}$ where $d\vec{r} \equiv (dx, dy, dz)$.

In physics, we can identify dx with an arbitrary increase in the magnitude x, with the condition that it is of the same order or smaller than the error ε_x, with which we measure x: $dx \lesssim \varepsilon_x$. The same can be stated for dy, dz and dt. We will also have $dU \lesssim \varepsilon_u$ (see the subsection *Error propagation* in Section 1.5).

Integral operations with vectors

There are three basic integral operations relating scalars and vectors.

The **temporal integral** of a scalar $U(\vec{r}, \dot{\vec{r}}, t)$ defined on a trajectory $\vec{r}(t)$ is the scalar

$$\int U dt = \int U(\vec{r}(t), \dot{\vec{r}}(t), t) dt \tag{1.23}$$

The **temporal integral** of a vector $\vec{V}$ defined on a trajectory $\vec{r}(t)$ is the vector

$$\int \vec{V} dt = \left(\int V_x dt, \int V_y dt, \int V_z dt \right) \tag{1.24}$$

The integral of a space defined vector $\vec{F}$ on a path C, which is also called ***circulation of $\vec{F}$ along*** C, if the path is given parametrically by $\vec{r}(\lambda) = (x(\lambda), y(\lambda), z(\lambda))$ from λ_1 to λ_2 (see Figure 1.10), is the scalar

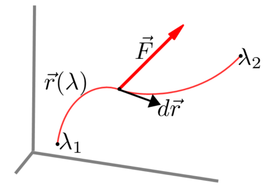


Fig. 1.10: Circulation of $\vec{F}$ along a path C

$$\int_C \vec{F} \cdot d\vec{r} = \int_{\lambda_1}^{\lambda_2} \vec{F} \cdot \frac{d\vec{r}}{d\lambda} d\lambda = \int_{\lambda_1}^{\lambda_2} \left(F_x \frac{dx}{d\lambda} + F_y \frac{dy}{d\lambda} + F_z \frac{dz}{d\lambda} \right) d\lambda \tag{1.25}$$

Problem 1.3.1. Given the vectors $\vec{A} = 5\hat{\imath} + 4\hat{\jmath} + 3\hat{k}$ and $\vec{B} = -2\hat{\jmath} + \hat{k}$, calculate:

a) The modules.

b) The scalar and vector products

c) The angle between them

d) Find a unit vector perpendicular to $\vec{A}$ and $\vec{B}$

Solution

a)

$$A = \sqrt{\vec{A} \cdot \vec{A}} = \sqrt{A_x A_x + A_y A_y + A_z A_z} = \sqrt{5^2 + 4^2 + 3^2} = 5\sqrt{2}$$

$$B = \sqrt{\vec{B} \cdot \vec{B}} = \sqrt{B_x B_x + B_y B_y + B_z B_z} = \sqrt{0^2 + (-2)^2 + 1^2} = \sqrt{5}$$

b)

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z = 5 \cdot 0 + 4 \cdot (-2) + 3 \cdot 1 = -5$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = 10\hat{\imath} - 5\hat{\jmath} - 10\hat{k}$$

c) From the scalar product, we get the angle between the vectors

$$\vec{A} \cdot \vec{B} = AB \cos\theta \Rightarrow \cos\theta = \frac{\vec{A} \cdot \vec{B}}{AB} = -\frac{1}{\sqrt{10}} \Rightarrow \theta = 108.4°$$

d) The vector product gives the expressions of a vector $\vec{C} = \vec{A} \times \vec{B}$ perpendicular to $\vec{A}$ and $\vec{B}$

$$\hat{C} = \frac{\vec{C}}{C} = \frac{\vec{A} \times \vec{B}}{\left|\vec{A} \times \vec{B}\right|} = \frac{10\hat{\imath} - 5\hat{\jmath} - 10\hat{k}}{\sqrt{10^2 + (-5)^2 + (-10)^2}} = \frac{2}{3}\hat{\imath} - \frac{1}{3}\hat{\jmath} - \frac{2}{3}\hat{k}$$ ■

Problem 1.3.2. Let $\vec{A}(t)$ be a vector that is a function of time. Prove that

$$\vec{A} \cdot \frac{d\vec{A}}{dt} = A\frac{dA}{dt}$$

and that if $\vec{A}$ has a constant modulus, then it is perpendicular to $\frac{d\vec{A}}{dt}$.

Solution

$A^2 = AA = \vec{A} \cdot \vec{A}$

$$\frac{d(A \cdot A)}{dt} = 2A\frac{dA}{dt}$$

$$\frac{d\left(\vec{A} \cdot \vec{A}\right)}{dt} = 2\vec{A} \cdot \frac{d\vec{A}}{dt}$$

$$\Rightarrow \vec{A} \cdot \frac{d\vec{A}}{dt} = A\frac{dA}{dt}$$

if $\vec{A}$ has a constant modulus, $\frac{dA}{dt} = 0, \Rightarrow \vec{A} \cdot \frac{d\vec{A}}{dt} = 0$, and therefore these vectors are perpendicular. ■

Fig. 1.11: Pierre Curie (1859-1906) was a French physicist

1.4 Principle of symmetry

The idea of symmetry is not exclusive to mechanics, nor even to physics. It has been used in many fields of science. *Reasons of symmetry* or *arguments of symmetry* are given to justify considerations that greatly simplify some problems, but it was not until relatively recently that any attempt has been made to specify what we mean by "reason of symmetry". Pierre Curie was the first to state a principle of symmetry, in 1894:

> An effect cannot lack a symmetry if this symmetry is not lacking in the cause.

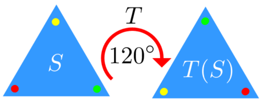


Fig. 1.12: Were it not for the vertices being labelled, the two triangles would be indistinguishable

That is, if the cause has a symmetry, it must be present in the effect. We will give a more detailed version of this idea, first by specifying what symmetry is.

With an object S and one transformation T of any kind (see Figure 1.12), we say that T is a symmetry transformation of S or that S has the T symmetry if S remains unchanged when applying T

$$T(S) = S \tag{1.26}$$

➔ ***Principle of symmetry:*** *If the causes have symmetries, then the effects have at least the same symmetries.*

Note that we need to make sure that we control all the causes, because it could happen that some of the causes are symmetrical and we have no information about the rest. In this case, the effects would not have to be identical! We use this principle many times without being aware of it, because it is very intuitive. Other times it does not seem like much to us, but it can also be applicable.

Problem 1.4.1. A uniform spherical mass distribution creates a gravitational field. Arguing exclusively with the symmetry principle, what can we say about the gravitational field?

Solution

The cause, that is, the uniform spherical mass distribution, has spherical symmetry: If we make any rotation around its centre, it remains unchanged. The effect, in this case the gravitational field, must therefore also have spherical symmetry. The gravitational field must be radial and depends only on the distance to the centre of the distribution.

■

1.5 Measurement and treatment of experimental data

We can never give a result in the form of a real number. This is so for many reasons. For example, suppose we want to measure the length of a rod like the one in Figure 1.13.

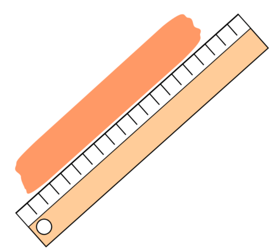


Fig. 1.13: The error in the measurement may be inherent to the measured object

1) The assumption that the rod has a well-defined length is an idealization. Real rods are not a sharply cut prism.

2) The measuring device has some limitations due to construction. No matter how well built the ruler is, it has divisions with a certain thickness that we can see. These divisions are separated in such a way that they coincide with the end of the rod by chance.

3) Furthermore, our eyes have some limitations that may vary according to lighting, age, etc.

Thus, the result of a measurement is not a real number; it is an interval. We can say, for example, that the length of the rod is within the range $[27.30, 27.40]$ cm (if the ruler has divisions up to mm). Otherwise expressed, we have 27.35 ± 0.05 cm. This same thing can be expressed as 27.3 cm, assuming that the number following the last one on the right can vary by one unit. In other words, we are working with an interval of $[27.30, 27.40]$ cm.

The width of the interval is given by the so-called ***measurement errors***. Clearly, the word *error* should not be viewed as something negative.

➔ A large measurement error does not have to reflect any kind of incompetence on the part of the person doing the measuring!

Error types

In order to give a reliable result of a measurement x, what we will do is repeat it M times $x_1,\ x_2,\ x_3 \dots x_M$. We will discard those values that clearly deviate from the majority and then be left with N good measurements, $x_1,\ x_2,\ x_3 \dots x_N$. We will express the result of the measurement as

$$x = \bar{x} \pm \varepsilon \tag{1.27}$$

with

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \tag{1.28}$$

where $\bar{x}$ is the average of the measurements and the number ε, positive, is the measurement error, which will generally be rounded up to one significant figure (this will be explained later). Depending on the causes, the error is decomposed into

$$\varepsilon = \varepsilon_I + \varepsilon_P \tag{1.29}$$

Innaccurate or systematic errors, ε_I: These are usually due to the measuring device itself. Whenever we make a measurement, commit a systematic error, which always has the same value.

Analog devices: Two examples of these are a needle moving on a scale and a ruler (see Figure 1.14), devices that give a continuous response. In this case, $\varepsilon_I = R/2$, where R is the resolution of the measuring device, otherwise known as the interval between marks. In a typical ruler, $R = 1\,\text{mm}$.

Fig. 1.14: Vernier calliper with analogical output

Digitals devices: These give a response at discrete intervals (typically, they have a numerical display; see Figure 1.15). In this case, $\varepsilon_I = R$ and matches the minimum increment shown by the device.

Fig. 1.15: Vernier calliper with digital output

Uncertain or probable errors ε_P: These are due to random factors, such as in the example of the rod in Figure 1.13, where the rod is not clearly cut at its ends and the measured length depends on where we place the ruler. There may also be environmental factors such as temperature, pressure, etc. The assumption that all measurements are made around a certain value (Gaussian distribution), together with the statistical treatment of the set of N measurements, which we will not detail here, gives a value for ε_P:

$$\varepsilon_P = \sigma\ f \tag{1.30}$$

where

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(\bar{x}-x_i)^2} \qquad (1.31)$$

is the **standard deviation** and

$$f = \frac{t}{\sqrt{N}} \qquad (1.32)$$

where t of (1.32) is the **statistical function of Student.**. Thus, f is a function that depends on the number N of performed measurements and, through the Student's t, depends also on the probability we want for ensuring that the actual value is within the given error range. If we are working with a probability of 95%, f is obtained according to Table 1.1.

N	f	N	f	N	f	N	f
		11	0.6718	21	0.4552	40	0.3198
2	8.984	12	0.6354	22	0.4438	60	0.2583
3	2.484	13	0.6043	23	0.4324	120	0.1808
4	1.592	14	0.5775	24	0.4223	$\to\infty$	$\to 0$
5	1.241	15	0.5538	25	0.4128		
6	1.050	16	0.5329	26	0.4039		
7	0.9248	17	0.5142	27	0.3956		
8	0.8360	18	0.4973	28	0.3876		
9	0.7687	19	0.4820	29	0.3804		
10	0.7154	20	0.4680	30	0.3734		

Table 1.1: Values of f as a function of the number of measurements N for a probability of 95%: $f = \frac{t_{0.975;N-1}}{\sqrt{N}}$

Note: The typical scientific calculators (see Figure1.16) and/or spreadsheets have procedures for entering data $x_1,\ x_2,\ x_3 \ldots x_N$ and for calculating $\bar{x}$ and σ. If necessary, read your calculator and/or spreadsheet manual.

Fig. 1.16: Read your calculator manual

Error propagation

In the previous sections, we have seen how to obtain and express a variable subject to error from its direct measurements. There are situations in which the same has to be done with variables that depend on other directly measurable variables. For example, if we have a variable z which depends on several variables $x, y \ldots$: $z = z(x, y \ldots)$, we can identify small errors with differentials, i.e., $\varepsilon_z = dz$, $\varepsilon_x = dx$, $\varepsilon_y = dy$..., etc. Because we know that $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy + \ldots$ we just need to evaluate the different factors $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, etc, with the mean values $x = \bar{x}$, $y = \bar{y}$, etc, and always take the absolute value (positive). We must also always use the different errors ε_x, ε_y, etc. Thus we obtain

$$z = \bar{z} \pm \varepsilon_z \quad ; \quad \bar{z} = z(\bar{x}, \bar{y}) \qquad (1.33)$$

with

$$\varepsilon_z = \left|\frac{\partial z}{\partial x}\right|_{\substack{x=\bar{x}\\ y=\bar{y}}} \varepsilon_x + \left|\frac{\partial z}{\partial y}\right|_{\substack{x=\bar{x}\\ y=\bar{y}}} \varepsilon_y + ... \tag{1.34}$$

Significant figures in direct measurements

In many cases, we do not need to refine the precision so much. It is enough to express the result of a measure with an interval width of $10^n \geq 2\varepsilon$, n being the minimum integer that satisfies the inequality. Returning to the case of the ruler and the rod of the Figure 1.13 as an example, the result of the measurement in cm is an interval [27.300000..., 27.399999...] cm.

We can say that 27.3 is the result given with one decimal place, but be careful! If instead of working in cm we do so in m, we will have 0.273, we would say that the result is given with three decimal places. If we were to do it in mm, the result would be 273 without decimal places. It should be clear that the expressions “with some decimal places” or “without decimal places” do not make much sense if we want to reflect how much precision we are working with. It is better to talk about significant figures: 2.73; 273; 0.000273; 273000. These are the results of measurements (intervals) with three significant figures. Let us look at some rules for detecting how many significant figures a given interval has.

Instead of talking about intervals or the result of a measurement, we will talk about numbers, following the usual practice. However, we must remember that they are not strictly numbers in the same way as natural, integers or real numbers, but are instead intervals expressed as numbers.

Example	SF
4.523	4
70.054	5
0.0789	3
0.0020	2
3600	2+?
3.6×10^3	2
3.60×10^3	3
3.600×10^3	4

Table 1.2: Examples of significant figures (SF)

→ **Rule 1**: In numbers that do not contain zeros, all digits are significant.

→ **Rule 2**: All zeros between significant digits are significant.

→ **Rule 3**: Zeros to the left of the first non-zero digit are only used to set the position of the decimal point and they are not significant.

→ **Rule 4**: In a number with digits to the right of the decimal point, zeros to the right of the last non-zero digit are significant.

→ **Rule 5**: In a number that has no decimal point and ends in zeros, the zeros may or may not be significant.

See the examples in Table 1.2.

Significant figures in indirect measurements

How are the results of the measurements added, subtracted, multiplied, and otherwise calculated? Clearly, it is done more or less as with the usual numbers. But what do we do with the significant figures? How many significant figures can be obtained from addition, subtraction, etc?

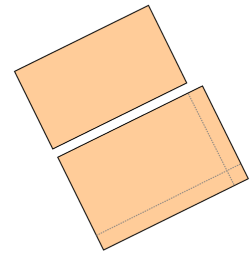


Fig. 1.17: Indirect measurement of the area of the rectangular sheet

Let us give a specific example. Suppose we have measured the sides of a (supposedly) rectangular sheet and want to know its area (see Figure 1.17). Let us say that the sides are $a = 5237$ mm and $b = 325$ mm. How do we express the result in order to correctly reflect the error made in the indirect measurement of the area? Is 1702025 mm^{2}correct? In other words, can we use seven significant figures? In this example, if the error is 1 mm per side, the error we commit for the area is on the order of $5237 \times 1 + 325 \times 1 + 1 \times 1 = 5563 \approx 6000$ mm^2.

The width of this range affects the digit in red: 1702025 mm^2. The correct procedure is, therefore, to express the result as 1.70×10^6mm^2, that is, with three significant figures.

The following general rules can be deduced for the manipulation of significant figures.

➔ **The mathematical constants** $(1/2, 2/3, \sqrt{2}, \pi ...)$: As long as they are taken with a higher number of digits than other the numbers involved, they do not affect the calculation of significant figures.

➔ **Multiplication and division:** The result of the operation will have the same number of significant figures as that of the operand with fewer significant figures.

➔ **Addition and subtraction:** The result must not have digits beyond the position of the last digit that is common to all addends. For example, $34.6 + 85 - 17.8 = 101.8$ should be rounded up to 102, as the position of the last digit common to all the terms is in the units.

Adjusting a line to the experimental data

Another very common situation is one that involves finding possible dependence between two or more variables that can be experimentally measured independently of each other, the values of which can be represented graphically. Through this representation, a possible relationship between the variables can be deduced. In the following, the methodology for determining this graph will be described.

➔ The representation can be made with a computer or manually (in the latter case, it is convenient to use graph paper).

➔ It is necessary to choose the axis scales while taking into account the extreme values of X and Y. This is done in such a way that the graph occupies a large part of the sheet of paper and the represented measurements are, as far as possible, evenly distributed (they do not accumulate at one extreme, for example).

➔ The point of intersection of the axes is not necessarily (0.0); it can be another point that makes the representation easy (**scale with displaced zero**).

➔ All obtained points must be drawn, without excluding any of them a priori. The data coordinates should not be written on either the axes or near each point, as this makes it difficult to observe the graph.

➔ The curve resulting from the adjustment of the experimental points must be continuous. Experimental points should never be joined with straight sections.

➔ If a point is clearly separated from the curve, it should not be taken into account in the adjustment and should be marked as erroneous.

➔ All graphics must be accompanied by an explanatory footnote. If several symbols are used for the different measured variables, it is mandatory to indicate which symbol corresponds to each variable.

The simplest case is that of two variables x, y, between which there may be a linear dependence. The straight line that connects them is called ***the line of fit*** and the most common procedure for obtaining it is ***least squares linear regression***. In this situation, we have a series of N measurements $x_1, x_2 ... x_N$ of a variable x; N measurements $y_1, y_2 ... y_N$ of a variable y; and we "suspect" that they are approximately related by a linear function $y = ax + b$. The a and b coefficients, as well as the goodness of fit, are determined by means of the **least squares method**, according to which the values of these parameters correspond to a straight line that minimizes the sum of the quadratic deviations χ^2 between the point of measurement y_i and the value $(ax_i + b)$ that takes the function for the variable x_i:

$$\chi^2(a,b) = \sum_{i=1}^{N} [y_i - (ax_i + b)]^2 \tag{1.35}$$

From the minimization process, we have

$$a = \frac{\overline{xy} - \overline{x}\,\overline{y}}{\overline{x^2} - \overline{x}^2} \qquad\qquad b = \overline{y} - a\,\overline{x} \tag{1.36}$$

where

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \;\; ; \;\; \bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i \;\; ; \;\; \overline{x^2} = \frac{1}{N}\sum_{i=1}^{N} x_i^2 \;\; ; \;\; \overline{xy} = \frac{1}{N}\sum_{i=1}^{N} x_i y_i \tag{1.37}$$

We can fit any set of points to a straight line. However, an index called the *correlation coefficient* $|r| \leq 1$, tells us the goodness of fit. This coefficient can be calculated according to the following expression:

$$r = \frac{\overline{xy} - \overline{x}\,\overline{y}}{\sqrt{\overline{x^2} - \overline{x}^2}\sqrt{\overline{y^2} - \overline{y}^2}} \tag{1.38}$$

The closer the points fit to the linearity hypothesis, the closer to 1 is r^2. In practice, a linear fit is considered good when $|r| \geq 0.95$ and cannot be linearly adjusted when $|r| < 0.8$, although these limits are debatable.

The probable errors and/or inaccuracies of the measurements of x and y cause random errors ε_a and ε_b of the coefficients a and b, and these can be evaluated using statistical methods:

$$\varepsilon_a = f\sigma_a \;\;;\;\; \varepsilon_b = f\sqrt{\overline{x^2}}\,\sigma_a \;\;;\;\; \sigma_a = \frac{a}{r}\sqrt{\frac{1-r^2}{N-2}} \tag{1.39}$$

where the f function is tabulated in Table 1.1 with the same interpretation (for 95%).

➔ **Note:** The usual scientific calculators (see Figure1.16) and/or spreadsheets have procedures for entering data $(x_1, y_1), (x_2, y_2) \ldots (x_N, y_N)$ and for calculating a, b and r . If necessary, read your calculator and/or spreadsheet manual.

Problem 1.5.1. In an experiment to find out the viscosity of castor oil, an aluminium sphere of density ρ_{Al} and radius R is released into a container with castor oil of density ρ_{Ol} and unknown viscosity η_{Ol}[2] . It is known that, after a certain time, the movement is approximately uniform with a terminal speed v_L given by the expression

$$v_L = \frac{P_{\text{ap}}}{b} \tag{1}$$

where P_{ap} is the apparent weight. Taking into account the Archimedean buoyant force (this studied in Section 4.3)

$$P_{\text{ap}} = \frac{4}{3}\pi R^3 g(\rho_{\text{Al}} - \rho_{\text{Ol}}) \tag{2}$$

and that b is the coefficient of viscous friction $\vec{F}_b$ (this is studied in Section 2.5)

$$\vec{F}_b = -b\vec{v} \tag{3}$$

b can be related to the radius of the sphere and the viscosity of the oil according to

$$b = 6\pi R\eta_{\text{Ol}} \tag{4}$$

The experiment consists of measuring the time it takes for the sphere to travel a certain distance z, counted after allowing the motion to approach a uniform motion.

[2] The castor oil used in this experiment has a viscosity of $0.985\,\mathrm{N\,s/m^2}$. It is this value that we want to determine through the described experience.

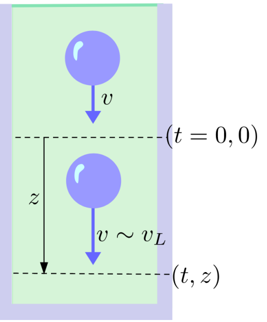


Figure for Problem 1.5.1

Data: $R = 0.02\,\text{m}$, $\rho_{\text{Al}} = 2700\,\text{kg/m}^3$, $\rho_{\text{Ol}} = 960\,\text{kg/m}^3$.

The results of the table were obtained from measurements made between $z = 0.25\,\text{m}$ and $z = 2.5\,\text{m}$

Table for Problem 1.5.1

z(m)	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50
t(s)	0.154	0.334	0.490	0.668	0.826	0.961	1.162	1.293	1.467	1.639

a) Represent the points (t, z) according to the measurements made.

b) Graphically find the straight line that best fits the points.

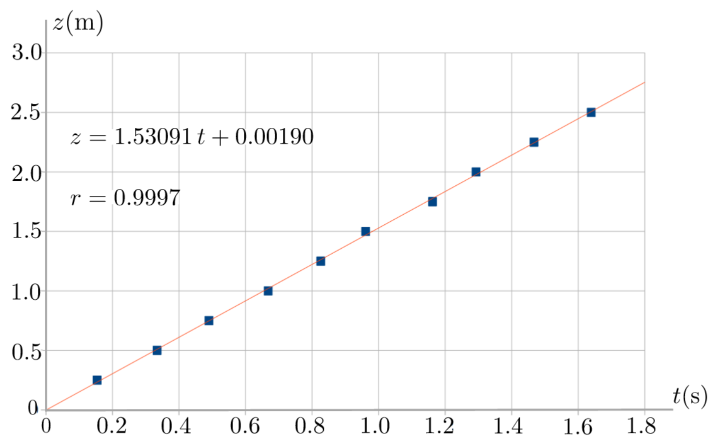
c) Find by **linear regression** the straight line that best fits these points. Justify the goodness of this fit.

d) Using the fitted straight line found in ***c)*** and the expressions (1, 2, 4), find v_L, b and η_{Ol}.

e) Find the slope error and the ordinate at the origin error of the fited straight line, and then calculate the propagation error in the viscosity η_{Ol}.

Solution

a), b) and **c)**. The represented points and the regression straight line can be seen in the figure. A spreadsheet has been used to do the linear regression. The analytic expression for the regression line gives: $z = A\,t + B = 1.53091\,t + 0.00190$, with a correlation coefficient of $r = 0.9997$



Solution to Problem 1.5.1

d) Relating the straight line and the fact that the movement is uniform according to $z = v_L t$, we obtain $v_L = 1.5309\,\text{m/s}$. Taking into account (2), $P_{\text{ap}} = 0.5720\,\text{N}$, and according to (1), we obtain $b = 0.3736\,\text{N s/m}$. By using (4), we have $\eta_{\text{Ol}} = 0.9911\,\text{N s/m}^2$.

e) To find the slope A and the ordinate at origin B errors, we use (1.39). With a number of measurements $N = 10$, we have $\sigma_A = 0.01326$ and, from Table 1.1, $f = 0.7154$. The obtained slope error is $\varepsilon_A = 0.0095\,\text{m/s}$, which will also be the error of v_L. Considering that, we can write

$$\eta_{\text{Ol}} = \frac{P_{\text{ap}}}{6\pi R} \frac{1}{v_L}$$

Given that the only magnitude subject to error on the right side is v_L, and taking into account (1.34), we have

$$\varepsilon_{\eta_{\text{Ol}}} = \frac{P_{\text{ap}}}{6\pi R} \frac{1}{v_L^2} \varepsilon_{v_L}$$

where $\varepsilon_{v_L} = \varepsilon_A = 0.0095\,\text{m/s}$. We obtain $\varepsilon_{\eta_{\text{Ol}}} = 0.0062\,\text{N s/m}^2$, and the resulting ordinate at the origin B error is $\varepsilon_B = 0.0097\,\text{m}$. Observe that this value makes the B value that we found by regression compatible with the value 0, which we expect according to the expression $z = v_L t$. Finally, we can express the result of the experiment according to the value found for η_{Ol}

$$\eta_{\text{Ol}} = (0.991 \pm 0.007)\,\text{N s/m}^2$$

■

1.6 Newton's first law. Inertial reference frames

Newton's first law

Newton's first law was motivated by the need to clarify what happens when nothing happens. In other words, the spacetime conditions in which agents move: the interacting particles and their corresponding forces. Note that in those days it was commonly believed that things did not move by themselves; yet, Galileo had already conceived that bodies have *inertia*, a natural tendency to maintain their state of motion.

Fig. 1.18: Isaac Newton (1643-1727) was an English physicist, mathematician and philosopher

➔ ***Newton's first law or the law of inertia**: A body unaffected by any cause moves at a constant velocity (uniform rectilinear motion).*

We can view this as defining the absence of cause: A body observed to be moving at a constant velocity is being affected by nothing. We can therefore say that it has **inertial motion**.

Fig. 1.19: Galileo Galilei (1564-1642) was a Tuscan physicist, mathematician and philosopher

Inertial reference frames

If an observer or reference frame moves inertially at a rectilinear and uniform speed, it is not affected by any external physical agent. How will he or she know how fast they are moving? Everything that is observed will seem to be a consequence of the same laws of physics that apply when not moving. This consideration invokes the definition of an inertial reference frame.

➔ **Inertial reference frames**: These are reference frames obtained by transforming a given reference frame in such a way that the laws of physics take the same form. The transformations from one to another are called ***inertial transformations*** (see Figure 1.20).

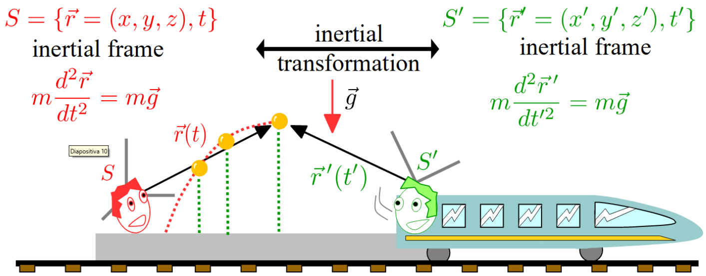


Fig. 1.20: Two inertial observers use the same laws of motion to explain their observations of the same event

Inertial transformations

Highly generalized spacetime hypotheses posit that two reference frames S and S' are inertial if they are related to each other by a composition of translation, rotation, change in the origin of time and a transformation of velocity. Both systems travel at a constant relative velocity $\vec{V}$. The expression of the latter depends on whether or not a maximum speed exists for synchronizing the different points in space.

➔ **Galileo transformations:** If nature does not limit the speeds, there are no problems with synchronizing. For the velocity transformation, we obtain

$$\begin{cases} \vec{r}' = \vec{r} - \vec{V}\,t \\ t' = t \end{cases} \tag{1.40}$$

These transformations define the Galilean inertial frames we will use to study Newtonian mechanics.

Fig. 1.21: Hendrik Antoon Lorentz (1853-1928) was a Dutch mathematician

➔ **Lorentz-Poincaré transformations**: If nature places a limit on speeds, we will have a **maximum speed** c that we use for synchronization. We therefore obtain

$$\begin{cases} \vec{r}' = \vec{r} - \gamma\vec{V}\,t + \frac{(\gamma-1)}{V^2}(\vec{V}\cdot\vec{r})\,\vec{V} \\ t' = \gamma t - \frac{1}{c^2}\gamma\vec{V}\cdot\vec{r} \end{cases} \tag{1.41}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$. Since c is the maximum speed there is no contradiction, so $V < c$ and the squareroot are always real. These transformations define the Poincaré inertial frames upon which Albert Einstein's special theory of relativity is based.

Fig. 1.22: Henri Poincaré (1854-1912) was a French mathematician

It is an experimental fact that there exists a maximum speed that coincides with that of light in a vacuum: $c = 299792458\,\mathrm{m/s}$. The consequences of this fact are known as the special theory of relativity, which has been experimentally proven. However, whenever we deal with systems that have velocities $v \ll c$ we can use the $c \to \infty$ approximation and Newtonian mechanics for our calculations. In Newtonian mechanics, time is absolute: For every point A, $t_A = t_O$; that is, all points in the space of an observer can be synchronized with the same time t. Unless explicitly stated otherwise, we will study Newtonian mechanics.

Regardless of whether or not we apply relativity, choosing an inertial reference frame for defining another one depends on the degree of approximation we are working with. In general, it is enough to consider the Earth as an inertial frame, although we know that it moves around the Sun and therefore is not strictly inertial. If more precision is needed, the inertial frame attached to the Sun can be used. For example, GPS (Global Positioning System) technology currently links its reference frames to quasars (*quasi-stellar radio sources*), which are extremely distant celestial objects.

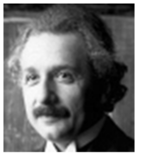


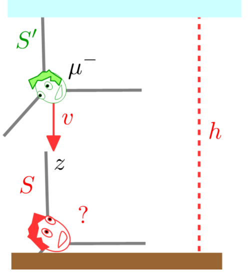
Fig. 1.23: Albert Einstein (1879-1955) was a physicist of German descent, later nationalized in Switzerland and the United States

To end this section, we will solve a single relativistic problem that draws on a specific and real case to illustrate how relativistic mechanics explained a fact that caused surprise at the time, because it could not be explained by Newtonian mechanics.

Problem 1.6.1. In the laboratory, low speed μ^- muons have a half life of $\tau = 2.197 \times 10^{-6}$ s. Cosmic rays reach the atmosphere at a height of $h = 2.5 \times 10^6$ m and produce μ^-, which are detected on the Earth's surface. Calculate in the following two ways the (constant) speed at which muons reach Earth.

a) without relativistic mechanics.

b) using relativistic mechanics.



Solution to Problem 1.6.1

Solution

a) Time is the same for all observers. The μ^- can "live", at most, $\tau = 2.197 \times 10^{-6}$ s. According to the S reference frame, at rest on the ground, μ^- should travel at a minimum speed $v = \frac{h}{\tau} = \frac{2.5\times10^6}{2.197\times10^{-6}} = 1.138 \times 10^{12}$ m/s, which far exceeds the maximum possible speed!

b) Here, we give the relativistic explanation. Time depends on the observer. The time to consider for the muon μ^- is its proper time, which in this case is the time of the reference frame S' that is traveling with the muon. With this interpretation, the muon can live at most $\Delta t' = \tau = 2.197 \times 10^{-6}$ s , which is the time measured by a clock at rest relative to S'. In contrast, to measure the speed of the muon, v, the observer S uses his or her time Δt, $v = \frac{h}{\Delta t}$.

Taking increments to the relativistic Lorentz-Poincaré transformation of time (1.41), with $\vec{V} = \vec{v}$, $\vec{v} \cdot \vec{r} = vz$, $\Delta z = h$ and $\frac{h}{\Delta t} = v$, we have

$$\Delta t' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \Delta t - \frac{1}{c^2} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} vh$$

Dividing everything by h and simplifying, we have

$$\frac{\Delta t'}{h} = \frac{1}{v}\sqrt{1 - \frac{v^2}{c^2}}$$

Therefore, the minimum speed v at which the muon must travel in order to reach the ground alive must satisfy

$$\left(\frac{v}{c}\right)^2 = \frac{1}{1+\left(\frac{c\Delta t'}{h}\right)^2} = 0.9999$$

that is, $v < c$, which is consistent with the fact that c is the maximum possible speed and, therefore, is consistent with relativity. ■

1.7 Point kinematics: position, trajectory, velocity and acceleration

➔ **Position vector:** The position vector of point $P = (x_P, y_P, z_P)$ relative to point $O = (x_O, y_O, z_O)$ is $\vec{r}_{P(O)} = (x_P - x_O, y_P - y_O, z_P - z_O)$. If $O = (0,0,0)$, it is written as $\vec{r}_P = (x_P, y_P, z_P)$ or, if there is no doubt, $\vec{r} = (x, y, z)$ (see Figure 1.24). Generally, we can write $\vec{r}_{P(Q)} = \vec{r}_P - \vec{r}_Q$.

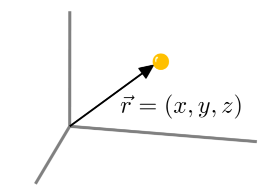


Fig. 1.24: Position vector with respect to the origin

In general, two observers will assign different position vectors for a given point P: $\vec{r}_{P(O)} \neq \vec{r}_{P(O')}$

➔ **Position vector module:** We denote this by $|\vec{r}| = r$.

➔ **Trajectory:** The trajectory of a particle is the curve $\vec{r}(\lambda)$ that it follows.

➔ **Temporal trajectory:** The temporal trajectory of a particle is the time parametrized curve $\vec{r}(t)$ that it follows (see Figure 1.25).

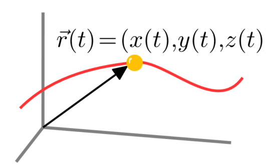


Fig. 1.25: Temporal trajectory of a particle

➔ **Displacement vector:** The displacement vector between two points is $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$. If the two points are infinitely close, then $d\vec{r} = (dx, dy, dz)$.

➔ **Module of $d\vec{r}$:** Be careful to note that dr is not the module of $d\vec{r}$, which we denote as $d\ell$, specifically $|d\vec{r}| = d\ell$, where ℓ is the length of the curve relative to some reference point on it (see Figure 1.26). It is only when the displacement $d\vec{r}$ is aligned with $\vec{r}$ that $d\ell = dr$ is fulfilled.

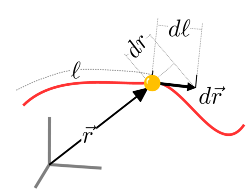


Fig. 1.26: Length of the curve ℓ, module of $d\vec{r}$ and dr

➔ **Differential of U:** Given a scalar $U(\vec{r})$ the differential of U, is $dU = \vec{\nabla}U \cdot d\vec{r}$ and it represents the infinitesimal variation of the U function when the position varies infinitesimally in a specified direction of the space $d\vec{r}$. If U explicitly depends on the time, $U(\vec{r}, t)$, and the position varies infinitesimally in an unspecified space direction $d\vec{r}$ spending an unspecified time dt the U variation is

$$dU = \vec{\nabla}U \cdot d\vec{r} + \frac{\partial U}{\partial t} dt \tag{1.42}$$

➔ **Velocity vector:** The particle velocity vector is the tangent vector to the trajectory, and its module indicates the rate of change of the particle position at every instant:

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) \tag{1.43}$$

➔ **Relative velocity:** The velocity of one P particle relative to another Q is:

$$\vec{v}_{P(Q)} = \frac{d\vec{r}_{P(Q)}}{dt} = \vec{v}_P - \vec{v}_Q \tag{1.44}$$

➔ **Acceleration vector:** The acceleration vector of a particle is the vector

$$\vec{a} = \dot{\vec{v}} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right) \tag{1.45}$$

Frenet trihedron

The rate of change in velocity $\vec{v}$ of a particle is the acceleration $\vec{a} = \dot{\vec{v}}$. The Cartesian components of these vectors do not tell us much about what they represent. Returning to the definition of a vector as magnitude with modulus and direction, we can thus express the velocity vector with the form $\vec{v} = v\hat{v}$. In this way, each factor has a very clear physical meaning, with v being the modulus and $\hat{v}$ being the direction. Now, we can derive to find the acceleration using

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(v\ \hat{v})}{dt} = \frac{dv}{dt}\hat{v} + v\frac{d\hat{v}}{dt} \tag{1.46}$$

with the first term corresponding to acceleration in the velocity's direction. This is the tangent acceleration $\vec{a}_T = \frac{dv}{dt}\hat{v}$. The tangential component of the acceleration is a_T, $a_T = \frac{dv}{dt}$.

To study what the second term $v\frac{d\hat{v}}{dt}$ represents, note that $\hat{v}$ is a unit vector, $\hat{v}^2 = 1$. Deriving this with respect to time, we have $2\hat{v} \cdot \frac{d\hat{v}}{dt} = 0$, meaning that $\frac{d\hat{v}}{dt}$ is a vector normal to $\hat{v}$, that is, normal to the trajectory. The $\hat{n} = \frac{d\hat{v}}{dt} \left|\frac{d\hat{v}}{dt}\right|^{-1}$ vector is unitary and normal to the trajectory. We can still define a third vector, the binormal vector $\hat{b}$, which is normal to both $\hat{v}$ and $\hat{n}$ as well as unitary: $\hat{b} = \hat{v} \times \hat{n}$. We will not use this third vector to describe the acceleration but instead define it only to complete the basis. What we are interested in analysing is the part of the acceleration aligned with the normal direction $\hat{n}$. This is the normal acceleration $\vec{a}_N = v\left|\frac{d\hat{v}}{dt}\right|\hat{n}$. To adequately describe the normal component of acceleration, it is necessary to know more about the trajectory.

Figure 1.27 compares $\hat{v}$ of two points that are very close in the trajectory. These are $\hat{v}(t)$ and $\hat{v}(t + dt)$, which form a triangle with two sides of length 1 and a third of length $|d\hat{v}|$. Because this is an isosceles triangle with an infinitesimal angle, it is comparable to an arc of radius 1. At the same time, the small section of the trajectory is a curve shaped as a small arc with angle $d\phi$, **radius of curvature** R and an arc length of $d\ell = vdt$. We thus have the relationships

$$1\ d\phi = |d\hat{v}| \quad ; \quad R\ d\phi = d\ell \tag{1.47}$$

which, by eliminating $d\phi$, allow us to obtain an expression of the radius of curvature R in terms of the unit vector tangent to the trajectory $\hat{v}$. For convenience, the **curvature** is defined as $\rho = 1/R$. All in all, we have the following.

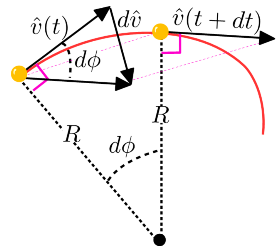


Fig. 1.27: Radius of curvature R

➔ **Radius of curvature:** At each point of a given trajectory with tangent unit vector $\hat{v}$, a radius of curvature R and a curvature ρ can be associated according to

$$\rho = \frac{1}{R} = \left|\frac{d\hat{v}}{d\ell}\right| \tag{1.48}$$

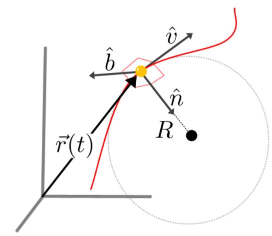


Fig. 1.28: Frenet trihedron and radius of curvature

➔ **Frenet trihedron:** At each point of a given trajectory with tangent unit vector $\hat{v}$, we can define a basis of vectors formed by the tangent, normal and binormal vectors to the trajectory called the *Frenet trihedron*; (see Figure 1.28) specifically according to

$$\hat{v} \quad ; \quad \hat{n} = R\frac{d\hat{v}}{d\ell} \quad ; \quad \hat{b} = \hat{v} \times \hat{n} \tag{1.49}$$

➔ **Tangent and normal acceleration:** The acceleration of a particle moving through a trajectory that is defined by the $\hat{v}$ unit vector can always be written as $\vec{a} = a_T\hat{v} + a_N\hat{n}$. The tangent a_T and normal a_N, which are acceleration components, can be expressed in terms of v and R as

$$a_T = \frac{dv}{dt} \quad ; \quad a_N = \frac{v^2}{R} \tag{1.50}$$


Fig. 1.29: Jean Frédéric Frenet (1816-1900) was a French mathematician, astronomer and meteorologist

Problem 1.7.1. An inertial observer O measures the position of a particle (SI units) $\vec{r}_{P(O)} = (6t^2 - 4t)\hat{\imath} - 3t^3\hat{\jmath} + 2\hat{k}$. Another observer O', with the same orientation, measures the position of the same particle $\vec{r}_{P(O')} = (6t^2 + 3t)\hat{\imath} - 3t^3\hat{\jmath} - 3\hat{k}$.

a) Determine the relative velocity of the reference frame O' with respect to O.

b) Calculate the particle acceleration with respect to O and O'.

c) Is O' an inertial observer?

Solution

a) The velocity of the particle with respect to O is:

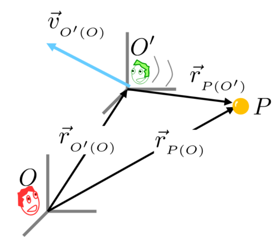
$$\vec{v}_{P(O)} = \frac{d\vec{r}_{P(O)}}{dt} = (12t-4)\hat{\imath} - 9t^2\hat{\jmath}$$

The velocity of the particle with respect to O' is:

$$\vec{v}_{P(O')} = \frac{d\vec{r}_{P(O')}}{dt} = (12t+3)\hat{\imath} - 9t^2\hat{\jmath}$$

The relative velocity of O' with respect to O will be

$$\vec{V} = \vec{v}_{O'(O)} = \vec{v}_{O'} - \vec{v}_O = -\vec{v}_P + \vec{v}_{O'} + \vec{v}_P - \vec{v}_O = \vec{v}_{P(O)} - \vec{v}_{P(O')} = -7\hat{\imath}$$



Solution to Problem 1.7.1

b)

$$\vec{a}_{P(O)} = \frac{d\vec{v}_{P(O)}}{dt} = 12\hat{\imath} - 18\,t\,\hat{\jmath}$$

From the previous section, we have $\vec{v}_{P(O')} = \vec{v}_{P(O)} - \vec{V}$, thus

$$\vec{a}_{P(O')} = \frac{d\vec{v}_{P(O')}}{dt} = \frac{d\vec{v}_{P(O)}}{dt} - \frac{d\vec{V}}{dt} = \vec{a}_{P(O)} = 12\hat{\imath} - 18\,t\,\hat{\jmath}$$

c) Yes, because O' is traveling at constant velocity with respect to an inertial observer O: $\vec{v}_{O'(O)} = \text{ct}$ ■

Problem 1.7.2. A particle describes a rectilinear motion, travelling in space $s = 4t^3 - 3t^2 - 6$, with s in meters and t in seconds. If the particle starts from $t = 0$, calculate

a) the time it will take to reach a speed of $6\,\text{m/s}$.

b) the value of its acceleration at the same instant.

Solution

$$v(t) = \frac{ds}{dt} = 12t^2 - 6t$$

a) If the particle starts from $t = 0$,

$$v(t) = 6 = 12t^2 - 6t \Rightarrow 2t^2 - t - 1 = 0 \Rightarrow t = 1s$$

b) At this moment, the acceleration will be

$$a(t=1) = \left.\frac{dv}{dt}\right|_{t=1} = (24t-6)|_{t=1} = 18\,\text{m/s}^2$$ ■

Problem 1.7.3. A body describes a rectilinear motion with acceleration $a = 4 - t^2$ (a in m/s^2 and t in s). Calculate the velocity and the displacement as a function of time, if at $t = 3\,\text{s}$, $v = 2\,\text{m/s}$ and $x = 9\,\text{m}$.

Solution

By integrating the expression of the acceleration we obtain the speed:

$$v = \int a dt = \int \left(4 - t^2\right) dt = 4t - \frac{t^3}{3} + K_1$$

Integrating this again we get the displacement

$$x = \int v dt = \int \left(4t - \frac{t^3}{3} + K_1\right) dt = 2t^2 - \frac{t^4}{12} + K_1 t + K_2$$

K_1 and K_2 are integration constants that depend on the initial conditions. With $t = 3\,\mathrm{s}$, $v = 2\,\mathrm{m/s}$ and $x = 9\,\mathrm{m}$, we have

$$v(3) = 2 = \left(4 \cdot 3 - \frac{3^3}{3} + K_1\right) \Rightarrow K_1 = -1$$

$$x(3) = 9 = \left(2 \cdot 3^2 - \frac{3^4}{12} + (-1)\,3 + K_2\right) m \Rightarrow K_2 = 0.75$$

Finally, we obtain

$$x = -t + 2t^2 - \frac{t^4}{12} + 0.75 \quad ; \quad v = 4t - \frac{t^3}{3} - 1$$

■

Problem 1.7.4. The three-dimensional motion of a particle is defined by the vector position $\vec{r} = R \sin(\omega t)\,\hat{\imath} + c\,t\,\hat{\jmath} + R\cos(\omega t)\,\hat{k}$ with R, ω and c constants.

a) Determine the particle's magnitudes of velocity and acceleration.

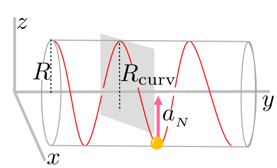
b) Calculate the radius of curvature, the tangent, the normal components of the acceleration and the unit normal vector.

Solution

a)

$$\vec{v} = \frac{d\vec{r}}{dt} = \omega R \cos \omega t \;\hat{\imath} + c \;\hat{\jmath} - \omega R \sin \omega t \;\hat{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = -\omega^2 R \sin \omega t \;\hat{\imath} - \omega^2 R \cos \omega t \;\hat{k} = -\omega^2 R \left(\sin \omega t \;\hat{\imath} + \cos \omega t \;\hat{k}\right)$$



Solution to Problem 1.7.4

b)

$$v = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{\omega^2 R^2 \left(\cos^2 \omega t + \sin^2 \omega t\right) + c^2} = \sqrt{\omega^2 R^2 + c^2}$$

$$a = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{\omega^4 R^2 \left(\sin^2 \omega t + \cos^2 \omega t\right)} = \omega^2 R$$

$$a_T = \frac{dv}{dt} = 0$$

With $a_T = 0$, we can conclude that the entire acceleration is normal, that is, $a_N = \omega^2 R$. If $a_T \neq 0$, we can solve for a_N using $a^2 = a_T^2 + a_N^2$. However, we will not do it in this way but will instead first solve it by finding the curvature:

$$\hat{v} = \frac{\vec{v}}{v} = \frac{\omega R \cos \omega t \;\hat{\imath} + c\hat{\jmath} - \omega R \sin \omega t \;\hat{k}}{\sqrt{\omega^2 R^2 + c^2}}$$

$$\frac{d\hat{v}}{dt} = \frac{-\omega^2 R \left(\sin\omega t\ \hat{\imath} + \cos\omega t\ \hat{k}\right)}{\sqrt{\omega^2 R^2 + c^2}}$$

$$\left|\frac{d\hat{v}}{dt}\right| = \frac{\omega^2 R}{\sqrt{\omega^2 R^2 + c^2}}$$

We will denote the radius of curvature by R_{curv} so as not to confuse it with R, which is a constant of the problem statement.

$$R_{\text{curv}} = v\left|\frac{d\hat{v}}{dt}\right|^{-1} = \frac{\sqrt{\omega^2 R^2 + c^2}}{\frac{\omega^2 R}{\sqrt{\omega^2 R^2 + c^2}}} = R + \frac{c^2}{\omega^2 R}$$

If $c = 0$ the curve does not advance in the $\hat{\jmath}$ direction, it is a circumference of radius R in the $x - z$ plane. In this case, $R_{\text{curv}} = R$.

We will calculate the normal acceleration a_N using the radius of curvature R_{curv}

$$a_N = \frac{v^2}{R_{\text{curv}}} = \frac{\omega^2 R^2 + c^2}{\left(\frac{\omega^2 R^2 + c^2}{\omega^2 R}\right)} = \omega^2 R$$

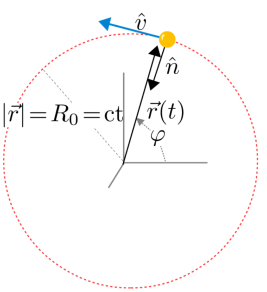
$$\hat{n} = \frac{R_{\text{curv}}}{v}\frac{d\hat{v}}{dt} = -\sin\omega t\ \hat{\imath} - \cos\omega t\ \hat{k}$$ ■

Problem 1.7.5. A particle describes a circular (non-uniform) motion $\vec{r}(t) = R_0(\cos\varphi, \sin\varphi, 0)$ where $\varphi = \varphi(t)$ is an increasing function of time. Calculate:

a) the velocity and the acceleration.

b) the trajectory's radius of curvature.

c) the tangent $\hat{v}$ and normal $\hat{n}$ basis vectors and the acceleration tangent a_T and normal a_N components.



Solution to Problem 1.7.5

Solution

a) The velocity and acceleration. When deriving with respect to t, we will take into account that we are doing a total derivative and φ is a function of t:

$$\vec{v} = R_0\dot{\varphi}(-\sin\varphi, \cos\varphi, 0)$$
$$\vec{a} = R_0\ddot{\varphi}(-\sin\varphi, \cos\varphi, 0) + R_0\dot{\varphi}^2(-\cos\varphi, -\sin\varphi, 0)$$

The velocity module is $v = R_0\dot{\varphi}$.

b) The $\hat{v}$ vector of the Frenet basis and the curvature radius R

$$\hat{v} = \frac{\vec{v}}{v} = (-\sin\varphi, \cos\varphi, 0) \quad ; \quad \frac{d\hat{v}}{dt} = \dot{\varphi}(-\cos\varphi, -\sin\varphi, 0)$$
$$R = v\left|\frac{d\hat{v}}{dt}\right|^{-1} = R_0$$

c) The $\hat{n}$ vector of the Frenet basis and the normal and tangent components of the acceleration

$$\hat{n} = \frac{R}{v}\frac{d\hat{v}}{dt} = (-\cos\varphi, -\sin\varphi, 0)$$
$$a_T = \frac{dv}{dt} = R_0\ddot{\varphi} \quad ; \quad a_N = \frac{v^2}{R} = R_0\dot{\varphi}^2$$ ■

→2

2 Dynamics of a particle

Introduction

We study the motion of a particle through the causes of its motion, that is, the forces. That is why we relate these causes to the concepts of kinematics (introduced in Chapter 1), such as position, velocity and acceleration. In the following, we will introduce other properties, such as momentum, moment of a force, angular momentum, work and energy. Finally, some types of forces will be analysed.

A **classical particle** is any material object whose dimensions are negligible in comparison to the dimensions of the trajectory that describes it (distances and radii of curvature). Furthermore, it either does not rotate on itself or, if it does, it does not transmit this rotation to any other particle. Extremely small particles of an atomic dimension or lower are excluded and dealt with by quantum mechanics. A **Newtonian particle** is a classical particle always moving at speeds much lower than that of light.

Unless otherwise stated, when we speak of a *particle* we mean Newtonian particle, as defined here. Also, when we say that an object is *small* what we want to say is that we can treat it as a Newtonian particle.

The concepts and methods explained in this chapter therefore apply to particles. However, later, in Section 3.13, we will see that if the forces acting on a rigid body do not rotate it, we can explain its motion by considering it as a particle located at a fixed point in relation to the body, which is called *centre of mass*. For the moment, however, we will talk about particles.

2.1 Newton's first and second laws

Newton's first law

Classical mechanics is based on three laws (i.e., axioms) established by Galileo and Newton in the 17^{th} and 18^{th} centuries, based largely on experimental results. In Section 1.6, we have already seen Newton's first law as it relates to defining an inertial frame and how it defines, among other things, what we mean by absence of cause. Here, we go further by specifying a little more the concept of cause. The first law can be formulated as:

➔ ***Newton's first law, or the law of inertia:*** *Every body (that is, a particle) remains at rest or maintains a uniform rectilinear motion unless it is acted upon by a* ***force****.*

Newton's first law introduces the concept of force as the cause that modifies a body's state of motion.

As indicated in Section 1.6, the reference frames (RF) that verify Newton's first law are called *inertial reference frames* (IRF). In addition, any RF moving at a constant velocity in relation to an IRF is also inertial. On the other hand, if an RF accelerates in relation to an IRF it is non-inertial. In this course, we will deal only with IRF and will at no time deal with *fictitious forces* such as centrifugal or Coriolis forces. We can solve all situations using an IRF without any need to introduce fictitious forces.

Newton's second law

The second law, also called *the law of motion*, states the following.

➔ ***Newton's second law, or the law of motion:*** *A body (that is, a particle) accelerates when subjected to the action of a non-zero force. The acceleration has the same direction as the force, and the modulus is equal to the force divided by the* ***inertial mass****. That is:*

$$\vec{a} = \frac{\vec{F}}{m} \tag{2.1}$$

$\vec{F}$ is the force originating the acceleration $\vec{a}$ of the particle. m is the proportionality constant between force and acceleration, is called *inertial mass* and is an intrinsic property of the particle that represents the resistance to be accelerated.

Much has been written about the meaning of this law and questions about it endure. Does it define force? Does it define (inertial) mass? Does the law help us to know how a particle moves? The truth is that it includes a little of everything.

It can be written as

$$\mathbf{A})\ \vec{F} = m\vec{a} \qquad \mathbf{B})\ \vec{F}(\vec{r},\vec{v},t) = m\vec{a} \qquad \mathbf{C})\vec{F}(\vec{r},\dot{\vec{r}},t) = m\ddot{\vec{r}} \tag{2.2}$$

Version **A** is the best known, as is (2.1), but it is not the most enlightening. The other two give more information and the third one, **C**, makes clear that we are referring to *the law of motion.*

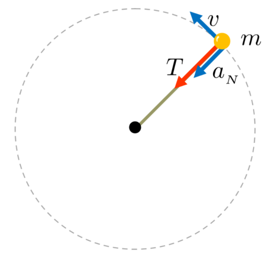


Fig. 2.1: The normal acceleration a_N is caused by the tension force T of the rope: $a_N = \frac{T}{m}$

➔ **Newton's law defines the force**: The force is the cause of the particle acceleration and it can depend on its position, velocity and time, but not on the acceleration. The force is a vector quantity, because it is the acceleration (as we have seen in Section 1.7) and the mass is scalar as it is an intrinsic property of the particle. In addition, $\vec{a}$ and $\vec{F}$ are parallel vectors and they have the same direction, due to m being positive.

It is important to note that (2.1) (and (2.2)) are vector equations. This means that they involve a module and a direction. Thus, using the example of a stone tied to a rope, acceleration occurs in the absence of gravity describing a circular trajectory while the velocity module remains constant. In this case, although the velocity module is constant, the direction is not. The acceleration is the normal (or centripetal) component, is directed towards the centre of the circular trajectory and originates in the tension of the rope, $a_N = \frac{T}{m}$ (see Figure 2.1 and Section 1.7.

➔ **Newton's law defines the mass:** If different forces $\vec{F}_1, \vec{F}_2$... act on the same particle, and cause different accelerations $\vec{a}_1, \vec{a}_2$... (see Figure 2.2), the relationship between the modules $\frac{F_1}{a_1} = \frac{F_2}{a_2} = ...$ remains constant and we will call it the ***inertial mass*** of the particle (or simply mass):

$$m = \frac{F_1}{a_1} = \frac{F_2}{a_2} = ... \tag{2.3}$$

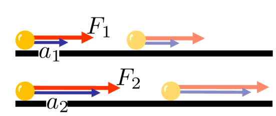


Fig. 2.2: Different forces on the same particle give rise to different accelerations. According to the Newton's second law, the quotient between each force and the corresponding acceleration modules remains constant and is called the *inertial mass* of the particle

The mass is a scalar quantity.

We must keep in mind that, in order to carry out the described experience, we first need a force pattern, whose definition does not include the concept of mass. This is not entirely straightforward and is based on applying the so-called **law of conservation of mass** to a classical and non-relativistic context: If we join a particle of mass m_1 with another particle of mass m_2, the mass of the new particle is $m_1 + m_2$. This law means that, classically, the mass can be used to measure the *amount of matter* and rename it as *law of conservations of matter*. To be highly precise, matter is measured by counting the number of either the protons, neutrons and electrons or the atoms and molecules that it contains.

One pattern we can use is the force near the Earth that receives a certain quantity of matter. Having done this, we can make a spring with a certain deformation that generates the same force as the one described above. For example, when this amount of matter is hung from the spring at rest, the force pattern will be generated by the spring. Now we can generate forces whose values are double, triple and more than the standard force using two, three and more springs arranged in parallel. If we use the superposition principle (which will be addressed later), the force will be double, triple, et cetera. These are the forces that we can use in the experiment described in the definition of mass. Obviously, the above-mentioned amount of matter becomes the standard unit of mass, and the unit of force is obtained according to Newton's law, as will be shown below.

→ **Newton's law defines the motion of a particle**. If we know the force as a field function of $\vec{r}$ and $\vec{v}$, the theory of differential equations tells us that **C** is an equation that can be integrated (twice) to find the unknown $\vec{r}(t)$. At each integration, a (vector) constant is introduced: $\vec{C}_1$ and $\vec{C}_2$. We will obtain $\vec{r}(t) = \vec{f}(\vec{C}_1, \vec{C}_2, t)$.

If at a given instant t_0 we know the position and the velocity of the particle, $\vec{r}_0 = \vec{r}(t_0)$ and $\vec{v}_0 = \vec{v}(t_0)$, we can solve

$$\begin{cases} \vec{r}_0 = \vec{f}(\vec{C}_1, \vec{C}_2, t_0) \\ \vec{v}_0 = \frac{\partial \vec{f}}{\partial t}(\vec{C}_1, \vec{C}_2, t_0) \end{cases} \tag{2.4}$$

and find $\vec{C}_1$ and $\vec{C}_2$ as a function of $\vec{r}_0$ and $\vec{v}_0$ thereby obtaining $\vec{r}(t) = \vec{r}(\vec{r}_0, \vec{v}_0, t)$.

→ **Predictive mechanics**. Newtonian mechanics is a predictive mechanics. This is a strong version of determinism. By knowing the force function acting on a particle, the position and the velocity at a given instant completely determine the particle's future and past.

In Section 2.5 of this chapter, some examples of forces will be seen.

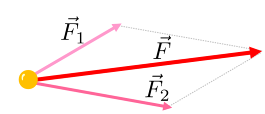


Fig. 2.3: Two forces acting on the same particle cause the same acceleration as a single force obtained by adding the two original forces

Principle of superposition of forces

Experiments show that if two forces $\vec{F}_1$ and $\vec{F}_2$ act on a particle, the acceleration $\vec{a}$ they cause on it is $\vec{a} = \vec{a}_1 + \vec{a}_2$, with $\vec{a}_i$, $i = 1, 2$, being the acceleration caused exclusively by force $\vec{F}_i$.

Taking into account Newton's law of motion, we can state the following (see Figure 2.3).

→ ***Principle of superposition of forces.*** *The resulting force from the two* $\vec{F}_1$ *and* $\vec{F}_2$ *forces acting on the same particle is*

$$\vec{F} = \vec{F}_1 + \vec{F}_2 \tag{2.5}$$

In order to fully appreciate this principle and not consider it trivial, note that this is not fulfilled in the theory of relativity!

Going back to Newton's first law, fulfilling it does not require that all the forces acting on a particle are zero; it is enough that its net force is null. The fact that the resulting force is zero does not imply absence of motion, only that the velocity is constant. At the time Newton devised his law, it shifted the paradigms followed by the Aristotelian school, which associated uniform motion with the existence of a force.

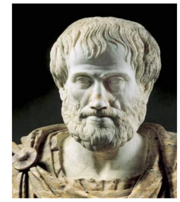


Fig. 2.4: Aristotle (384 bC-322 bC). Aristotelian physics: is the study of the being in motion. Between the laws of motion, states that "every mobile requires an engine"

Units of mass and force

Now all we have to do is introduce the mass pattern needed to make measurements.

➔ **The unit of mass in SI is the kilogram (kg):** A mass of 1 kilogram is equal to the mass of the international standard, which is an iridium-platinum cylinder guarded in Paris by BIPM (definition adopted in the year 1889).

The unit of force is derived from Newton's law.

➔ **The unit of force in SI is the newton (N):** $1\,\mathrm{N} = 1\mathrm{kg}\,\frac{\mathrm{m}}{\mathrm{s}^2}$

To do the following problems, be aware of the weight force or force due to the gravitational field $\vec{g}$ near the Earth's surface, $\vec{P} = m\vec{g}$, which acts on any m mass, with $g = 9.81\,\mathrm{m/s^2}$. If we want to be more precise, we will take into account the local gravity. For example, in Barcelona, $g = 9.804\,\mathrm{m/s^2}$. Weight will be treated as an example of constant force in Section 2.5 and weight force will be studied in more detail in Section 4.2.

Problem 2.1.1. A girl of mass 30 kg is in an elevator. Determine the force N exerted by the ground on the girl if the elevator:

a) rises with uniform motion.

b) descends with uniform motion.

c) rises with an acceleration of $2\,\mathrm{m/s^2}$.

d) descends with an acceleration $2\,\mathrm{m/s^2}$.

e) freefalls after the elevator cables break.

Solution

With $m = 30$ kg and $g = 9.81$ m/s^2, we use Newton's law of motion in the vertical direction, $N - mg = ma \Rightarrow N = m(a+g)$. Thus we obtain:

a) $a = 0 \Rightarrow N = mg =$ 294.3 N

b) $a = 0 \Rightarrow N = mg =$ 294.3 N

c) $a = 2 \Rightarrow N = m(2+g) =$ 354.3 N

d) $a = -2 \Rightarrow N = m(g-2) =$ 234.3 N

Solution to Problem 2.1.1

e) $a = -g \Rightarrow N = 0$ ■

Problem 2.1.2. A small object of $m = 4$ kg, is subject to the action of two forces, $\vec{F}_1 = \hat{\imath} - 2\hat{\jmath}$ and $\vec{F}_2 = \hat{\imath} + \hat{\jmath}$ (N units). Calculate the acceleration, velocity and position vectors of the object at time $t = 3$ s if at $t = 0$ it is at rest at the origin of coordinates.

Solution

$m = 4$ kg. The resulting force is $\vec{F} = \vec{F}_1 + \vec{F}_2 = (2, -1)$. From the equation of motion, we obtain the acceleration:

$$\vec{F} = m\vec{a} \Rightarrow \vec{a} = \frac{\vec{F}}{m} = \left(\frac{1}{2}, -\frac{1}{4}\right)$$

Integrating the acceleration,

$$\vec{v} = \int \vec{a}\, dt = \left(\frac{1}{2}, -\frac{1}{4}\right) t + \vec{C}_1$$

and taking into account the initial conditions at $t = 0$, $\vec{v} = 0$ implies $\vec{C}_1 = 0$ and, as a result, $\vec{v} = \left(\frac{1}{2}, -\frac{1}{4}\right) t$

Integrating the velocity,

$$\vec{r} = \int \vec{v}\, dt = \left(\frac{1}{4}, -\frac{1}{8}\right) t^2 + \vec{C}_2$$

and taken into account the initial conditions at $t = 0$, $\vec{r} = 0$ implies $\vec{C}_2 = 0$ and, as a result $\vec{r} = \left(\frac{1}{4}, -\frac{1}{8}\right) t^2$ ■

Problem 2.1.3. A particle tied to a rope of negligible mass and length ℓ describes a uniform circular motion. Find the minimum angular speed ω that makes it possible.

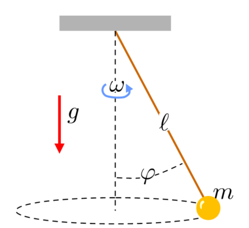


Figure for Problem 2.1.3

Solution

Newton's motion equations in vertical and normal path directions are:

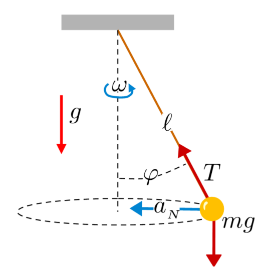
$$T\cos\varphi - mg = 0$$
$$T\sin\varphi = ma_N$$

The normal acceleration is $a_N = \frac{v^2}{R} = R\omega^2 = \ell\sin\varphi\,\omega^2$. Resolving T in the first and replacing it in the second, we have

$$\frac{mg}{\cos\varphi}\sin\varphi = m\ell\sin\varphi\,\omega^2$$

from which we get $\cos\varphi = \frac{g}{\ell\omega^2}$. The cosine is positive and $\cos\varphi \le 1$ must be fulfilled, thus

$$\omega \ge \sqrt{\frac{g}{\ell}}$$ ■



Solution to Problem 2.1.3

Problem 2.1.4. A particle tied to a rope of negligible mass and length ℓ oscillates in a vertical plane.

a) Find the equation of motion using Newton's second law.

b) Specify the result for small oscillations.

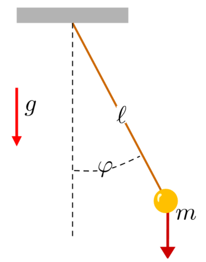


Figure for Problem 2.1.4

Solution

Newton's motion equations in normal and tangent path directions are:

$$T - mg\cos\varphi = ma_N$$
$$-mg\sin\varphi = ma_T$$

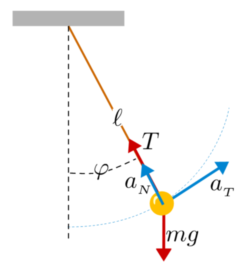
With the speed being $v = \ell\dot{\varphi}$, the tangent acceleration is $a_T = \frac{dv}{dt} = \ell\ddot{\varphi}$. Substituting the tangent component of the equation of motion $-mg\sin\varphi = m\ell\ddot{\varphi}$, we can thus simplify it to:

$$\ddot{\varphi} + \frac{g}{\ell}\sin\varphi = 0$$

If $\varphi \ll 1$, expressed in radians, then $\sin\varphi \approx \varphi$ and we obtain

$$\ddot{\varphi} + \frac{g}{\ell}\varphi = 0$$

■



Solution to Problem 2.1.4

2.2 Force and momentum

Momentum. The linear momentum or quantity of motion (or simply momentum, if there is no confusion), $\vec{p}$ is defined as the vector

$$\vec{p} = m\vec{v} \tag{2.6}$$

The SI unit of momentum is kg m/s.

Newton's law of motion can be written using the momentum as

$$\frac{d\vec{p}}{dt} = \vec{F} \tag{2.7}$$

Equation (2.7) allows interpreting the force as the cause of the momentum variation and thus states the following.

Conservation of momentum theorem. *If no forces act on a particle or their net force is zero, the momentum is constant.*

For a particle, this statement is immediate, because, in this case, the momentum is directly proportional to the velocity of the particle. We will see later in Section 3.3 where this concept is applied to N particles, that these expressions are also valid and that they will allow us to solve more complex situations.

Impulse applied by a force. The impulse $\vec{I}$ applied by a force $\vec{F}$ in the interval $t_1 \to t_2$ is defined as the vector

$$\vec{I} = \int_{t_1}^{t_2} \vec{F} dt \tag{2.8}$$

The impulse unit in SI is kg m/s. Note that in order to find $\vec{I}$, it may be necessary to know the trajectory $\vec{r}(t)$.

The momentum theorem. *The change in momentum is equal to the applied impulse* $\vec{I}$

$$\vec{I} = \Delta\vec{p} \tag{2.9}$$

Proof. One need only consider Newton's law (2.7) in the form $\vec{F}\,dt = d\vec{p}$ and take the integral in the interval $t_1 \to t_2$ ■

Problem 2.2.1. A 1 kg particle, for $t \leq 0$, moves rectilinearly at a constant speed of 100 m/s. At instant $t = 0$ and for a duration of 1 s a force $F = 1000\ \mathrm{e}^{-t}$ (F in N and t in s) acts in the same direction but opposite sign to the motion. Calculate the impulse applied by the force and the final momentum of the particle.

Solution

Since force and velocity have the same direction, the particle direction will not change. As we are working with a one-dimensional problem, the vector notation will be omitted. We have only two directions (let us say $\pm\hat{\imath}$). We take the positive sign as that of the initial velocity.

The impulse that the force supplies to the particle is

$$I = \int_{t_1}^{t_2} F dt = \int_0^1 -1000\,\mathrm{e}^{-t} dt = 1000\,\mathrm{e}^{-t}\Big|_0^1 = -632\,\frac{\mathrm{kg\ m}}{\mathrm{s}}$$

The final momentum p_2 is

$$\Delta p = p_2 - p_1 \Rightarrow p_2 = p_1 + \Delta p = mv_1 + \Delta p = 100 - 632$$
$$\Rightarrow p_2 = -532\,\mathrm{kg\ m/s}$$

This result means that the particle ends up moving at a constant speed of 532 m/s in a direction opposite to the initial velocity. ■

Problem 2.2.2. A particle of 2 kg of mass moves at a certain instant with a velocity expressed by $\vec{v} = 5\ \hat{\imath} + 2\ \hat{\jmath}$. Then, a force $\vec{F} = 4\ \hat{\jmath}$ is applied to it. Knowing that $\vec{v}$ and $\vec{F}$ are expressed in SI units, determine the momentum of the particle after applying the force for 3 s.

Solution

$m = 2$kg. The initial momentum is $\vec{p}_1 = m\vec{v} = 10\,\hat{\imath} + 4\,\hat{\jmath}$. By using the momentum theorem and being $\vec{p}_2$ the final momentum

$$\Delta\vec{p} = \vec{p}_2 - \vec{p}_1 = \int_t^{t+3} \vec{F}dt = 3\vec{F} = 12\,\hat{\jmath}$$

We note that the limits of integration, in principle, depend on time. The force is constant over time. Insulating $\vec{p}_2$, we obtain

$$\vec{p}_2 = \vec{p}_1 + 12\,\hat{\jmath} = 10\,\hat{\imath} + 16\,\hat{\jmath}$$

in SI units. ■

2.3 Torque and angular momentum of a particle

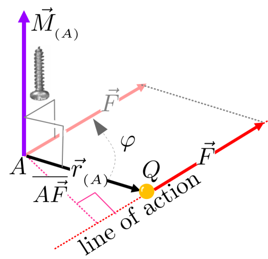


Fig. 2.5: Torque of force $\vec{F}$ with respect to point A. Corkscrew rule

➔ **Torque or moment of a force**. The torque $\vec{M}_{(A)}$ of a force $\vec{F}$ applied to a Q point with respecte point A is defined as

$$\vec{M}_{(A)} = \vec{r}_{(A)} \times \vec{F} \tag{2.10}$$

where $\vec{r}_{(A)}$ is the position vector of point Q, which is where the force is applied, relative to point A.

Note that a consequence of the vector product is that the direction of the torque of the force is perpendicular to the plane formed by the vectors $\vec{r}_{(A)}$ and $\vec{F}$. What is more, because the first vector of the (2.10) vector product is a position, the modulus can be calculated by means of the simple expression (see Figure 2.5):

$$M_{(A)} = F\ r_{(A)} \sin\varphi = F\ \overline{A\vec{F}} \tag{2.11}$$

where $\overline{A\vec{F}} = r_{(A)} \sin\varphi$ is also the distance between point A and the **line of action** of force a $\vec{F}$. The direction is given by the corkscrew rule.

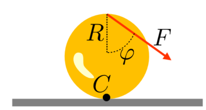
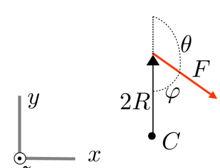


Figure for Problem 2.3.1

Problem 2.3.1. Calculate the torque of the $\vec{F}$ force applied to the sphere at the point indicated in the figure, with respect to the ground contact point C.

Solution



Solution to Problem 2.3.1

The torque can be obtained directly by the vector product. Looking at the figure, $\vec{r}_{(C)} = 2R(0,1,0)$ and $\vec{F} = F(\sin\varphi, -\cos\varphi, 0)$, and thus

$$\vec{M}_{(C)} = \vec{r}_{(C)} \times \vec{F} = 2RF \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 0 \\ \sin\varphi & -\cos\varphi & 0 \end{vmatrix} = -2RF\sin\varphi\,\hat{k}$$

We can also proceed by applying the corkscrew rule, which allows us to state $\vec{M}_{(C)} = -M_{(C)}\,\hat{k}$ and then calculate the modulus according to (2.11)

$$M_{(C)} = F\,\overline{C\vec{F}} = F\,2R\sin\varphi$$ ■

→ **Angular momentum**. The angular momentum of a particle with respect to point A, $\vec{L}_{(A)}$, is defined as the moment of momentum $\vec{p}$ applied to point Q,which is where the particle is (see Figure 2.6):

$$\vec{L}(A) = \vec{r}_{(A)} \times \vec{p} \tag{2.12}$$

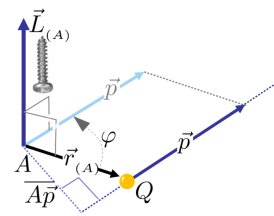


Fig. 2.6: Angular momentum $\vec{L}_{(A)}$. Corkscrew rule

where $\vec{r}_{(A)}$ is the position vector of point Q (where the momentum is applied) with respect to point A. It is important that A is a fixed point in the reference frame.

The unit of angular momentum in the SI is kg m^2/s.

Note that as a consequence of the vector product, the direction of angular momentum is perpendicular to the plane formed by the $\vec{r}_{(A)}$ and $\vec{p}$ vectors. What is more, because the first vector of the (2.12) vector product is a position, the modulus can be calculated by the simple expression:

$$L_{(A)} = p\ \overline{A\vec{p}} \tag{2.13}$$

where $\overline{A\vec{p}}$ is the distance between point A and the line of action of the $\vec{p}$ momentum. The direction is given by the corkscrew rule.

Newton's law of motion implies that:

$$\frac{d\vec{L}_{(A)}}{dt} = \vec{M}_{(A)} \tag{2.14}$$

Proof. Multiplying both members of Newton's law of motion $m\frac{d\,\vec{v}}{dt} = \vec{F}$ by $\vec{r}_{(A)}\times$ and taking into account that $\vec{r}_{(A)} \times \frac{d\vec{p}}{dt} = \frac{d}{dt}(\vec{r}_{(A)} \times \vec{p})$ (since $\dot{\vec{r}}_{(A)} = \vec{v}$ and $\vec{v} \times \vec{p} = 0$), we get the result. ■

We have shown Newton's law of motion implies (2.14), but not that they are equivalent: (2.14) **does not imply** Newton's law of motion.

Equation (2.14) allows us to interpret that the torque is the cause of the variation in the angular momentum, and we can thus state the following.

→ ***Conservation of angular momentum theorem.*** *If the torque of a force acting on a particle with respect to one point A is zero, the angular momentun relative to this point remains constant.*

We observe that both the angular momentum and the torque of a force depend on point A with respect where it is calculated. It is possible that relative to A we instead have $\vec{M}_{(A)} = 0$, and relative to another point B we have $\vec{M}_{(B)} \neq 0$.

The null torque of a force and, therefore, the conservation of the angular momentum occurs if the net force applied to the particle is zero. However, this also occurs in other more interesting cases, such as when the $\vec{r}_{(A)}$ vector and $\vec{F}$ force are parallel. That is why, even in the case of a single particle, we will use Equation (2.14) in order to solve some interesting problems.

As we will see later, the conservation of angular momentum theorem is especially important in the case of a particle system, particularly in rigid bodies.

➔ **Angular impulse**. The angular impulse $\vec{Y}$ applied by the torque $\vec{M}_{(A)}$ in the interval $t_1 \to t_2$ is defined as the vector

$$\vec{Y}_{(A)} = \int_{t_1}^{t_2} \vec{M}_{(A)} dt \tag{2.15}$$

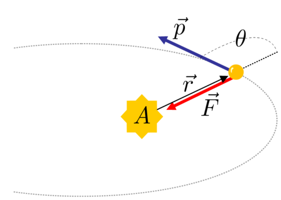
The unit of angular impulse in SI is $\mathrm{kg\,m^2/s}$.

➔ ***Angular momentum theorem***. *The increase in angular momentum is equal to the applied angular impulse $\vec{Y}$*

$$\vec{Y}_{(A)} = \Delta\vec{L}_{(A)} \tag{2.16}$$

Proof. One need only consider (2.14) in the form of $d\vec{L}_{(A)} = \vec{M}_{(A)} dt$ and integrate it into the interval $t_1 \to t_2$. ■

Problem 2.3.2. Planets are known to have elliptical trajectories relative to the Sun (which we consider to be fixed) at one foci. Prove Kepler's second law.

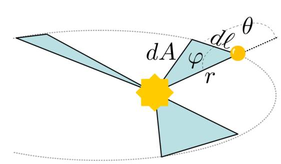


Solution to Problem 2.3.2

Solution

We take the torque with respect to the Sun (point A). The torque of the force is zero and, therefore, the angular momentum in conserved:

$$\vec{L} = \vec{r} \times \vec{p} = \mathrm{ct}$$



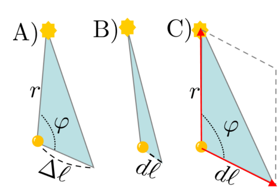
Solution to Problem 2.3.2

As the direction of angular momentum is fixed, the relevant information is contained only in the modulus

$$m\,r\,v \sin\theta = \mathrm{ct}$$

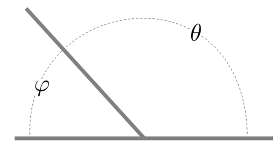
Taking in to account that the velocity modulus can be written as $v = \frac{d\ell}{dt}$, it becomes

$$r\,\frac{d\ell}{dt} \sin\theta = \mathrm{ct}$$



Solution to Problem 2.3.2

Looking at the figure, the areas swept by the position vector for a time are triangles that are, usually, scalene, such as **A** triangle in the figure. The length of the trajectory between the two vertices, $\Delta\ell$, does not match the corresponding side of the triangle. If we consider a time $\Delta t \to dt$, the planet will have travelled $\Delta\ell \to d\ell$ and we will have the **B** triangle in the figure. Now, the length of the trajectory between the two vertices, $d\ell$, coincides with the corresponding side of the triangle. We can expand it in order to see it better, such as with the **C** triangle in the figure. The area dA that is swept by the position vector in dt is thus equal to the area of the **C** triangle in the figure. Because we know that the modulus of the cross product of two vector is the area of the parallelogram they form, the area of our triangle will therefore be



Solution to Problem 2.3.2

$$dA = \frac{1}{2} r\, d\ell \sin\varphi = \frac{1}{2} r\, d\ell \sin\theta$$

Thus, comparing this result with that obtained from the conservation of angular momentum, we can conclude

$$\frac{dA}{dt} = \text{ct}$$

which is Kepler's second law: *The areas swept by the position vector of the planet at equal times are equal.* ■

Fig. 2.7: Johannes Kepler (1571-1630) was a German astronomer and mathematician

2.4 Work, kinetic energy and potential energy. Power

Work and energy are closely related concepts, that go beyond mechanics and play a key role in the world of physics.

Work done by a force

Let us look at the simple case of one particle moving rectilinearly over a distance Δx under the effects of a constant force F, parallel to and with the same direction as the displacement and perhaps in the presence of a frictional force (see Figure 2.8). The work done by the force F is defined by the product of the force F and the displacement Δx:

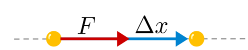


Fig. 2.8: One particle moves rectilinearly under the action of a force in the same direction as the displacement vector

$$W = F\,\Delta x \tag{2.17}$$

If the motion is rectilinear and the applied force is constant but forms an angle θ with respect to the displacement (see Figure 2.9), the work is:

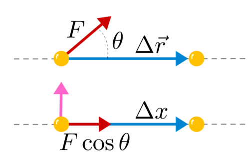


Fig. 2.9: One particle moves rectilinearly $\Delta\vec{r}$ under the action of a constant force $\vec{F}$. Only the component of the force in the direction of the displacement causes the particle to advance along the trajectory

$$W = F\,\Delta x \cos\theta = \vec{F}\cdot\Delta\vec{r} \tag{2.18}$$

Observe that in this case we consider only the component of the force in the direction of the displacement, while the perpendicular component performs no work.

In the general case where the force is not constant and the path C followed by the particle is not rectilinear (see Figure 2.10), calculating the work between two P_1 and P_2 points requires considering the infinitesimal displacements $d\vec{r}$ along the path, and all infinitessimal work $\vec{F}\cdot d\vec{r}$ must be added toghether.

➔ **Work done by a force**. We define the work W done by a force $\vec{F}$ along a path

C from P_1 to P_2 by means of the integral

$$W = \int_{C:P_1}^{P_2} \vec{F} \cdot d\vec{r} \tag{2.19}$$

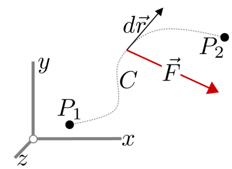


Fig. 2.10: Work done by a force $\vec{F}$ from P_1 to P_2 along the C path

If we know the parametric expressions of the path $\vec{r}(\lambda) = (x(\lambda), y(\lambda), z(\lambda))$, $P_1 = \vec{r}(\lambda_1)$ and $P_2 = \vec{r}(\lambda_2)$, we can make the integral work explicit:

$$W = \int_{\lambda_1}^{\lambda_2} \vec{F} \cdot \frac{d\vec{r}}{d\lambda} d\lambda = \int_{\lambda_1}^{\lambda_2} \left(F_x \frac{dx}{d\lambda} + F_y \frac{dy}{d\lambda} + F_z \frac{dz}{d\lambda} \right) d\lambda \tag{2.20}$$

The concept of work is highly important from both theoretical and practical point of view. The ingredients for calculating work are a force $\vec{F}$ and a path C with two points on it. This force and path need not be related.

The following two interpretations are highlighted below.

➔ **1)** If force $\vec{F}$ contributes to moving the particle along path C, then the path can be parameterized with time t. $\vec{r}(t)$ is the time path and, in the absence of other forces it, fulfils $\vec{F}(\vec{r}, \dot{\vec{r}}, t) = m\ddot{\vec{r}}$. This interpretation has eminently practical utility, as in this case work becomes a measure of the effectiveness of the force that displaces the particle. There may be a lot of force and little work (little displacement) or little force and a lot of work (a lot of displacement). The work thus approximates our everyday concept of *work*, not so much as a measure of *effort* but as a measure of efficiency in transforming something in the environment (in our case, the displacement of the particle). Later we will see the conservative forces that are especially effective in accordance with this interpretation.

➔ **2)** Path C does not necessarily have to be a solution to the equation of motion with the force $\vec{F}$. That is, if $\vec{r}(t)$ is a time trajectory passing through C and m is the mass of the particle to which force $\vec{F}$ is applied, it does not necessarily follow that $\vec{F}(\vec{r}, \dot{\vec{r}}, t) = m\ddot{\vec{r}}(t)$. If the force depends only on the position, $\vec{F}(\vec{r})$, we can consider different paths to calculate the work. We do not even need to account for the presence of the particle. This interpretation, as we will see, is of great theoretical interest.

The concept of kinetic energy is related to work in terms of interpretation **1**, above, as we will see in the following. Instead, the concept of potential energy will require the most abstract interpretation **2**. The concept of power can be defined as long as we know the temporal trajectory of the particle, as the interpretation **1**.

➔ **Power of a force**. The power exerted by a force $\vec{F}$ at each instant is defined as the rate of work done by the force on the particle:

$$P = \frac{dW}{dt} = \vec{F} \cdot \vec{v} \tag{2.21}$$

In general, the particle will be subjected to different forces. If we know the temporal trajectory caused by these forces, we can calculate the power of each one separately.

Kinetic energy

The kinetic energy of a particle of mass m moving at a velocity $\vec{v}$ is defined as the capacity to do work associated with the fact that the particle is in motion. Thus, if we combine the definition of work, $W = \int \vec{F} \cdot d\vec{r}$, with Newton's second law, $\vec{F} = m\vec{a}$, and the definition of velocity, $\vec{v} = \frac{d\vec{r}}{dt}$, from which we obtain $d\vec{r} = \vec{v}dt$, we can write

$$W = \int \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \vec{v}\; dt = \int m\, \vec{a} \cdot \vec{v}\, dt \tag{2.22}$$

Now, taking into account that $\vec{a} \cdot \vec{v}\; dt = \frac{d\vec{v}}{dt} \cdot \vec{v}dt = \frac{d}{dt}\left(\frac{1}{2}\vec{v}^2\right) dt = d\left(\frac{1}{2}\vec{v}^2\right)$, we obtain

$$W = \int \frac{d}{dt}\left(\frac{1}{2} m\, v^2\right) dt = \int d\left(\frac{1}{2} m\, v^2\right) \tag{2.23}$$

➔ **Kinetic energy**. The kinetic energy, E_c, of a particle of mass m, moving with a velocity $\vec{v}$ is defined as

$$E_c = \frac{1}{2} m\, v^2 \tag{2.24}$$

➔ ***Work-energy theorem***. *The work done by a force on a particle is equal to the change in its kinetic energy.*

$$W = \Delta E_c \tag{2.25}$$

Proof. It is enough to finish the final integral in (2.23) with the integration limits $\int_{\text{ini}}^{\text{fi}}$ and take into account the definition of kinetic energy (2.24): $\int_{\text{ini}}^{\text{fi}} d\left(\frac{1}{2} m\, v^2\right) = E_{c:\text{fi}} - E_{c:\text{ini}}$ ■

Potential energy

The potential energy associated with a particle subjected to a conservative force is the capacity of doing work and it is related with occupying a certain position in space.

Consider the simple case of a particle moving in one dimension (we will use the x coordinate) under the effects of a force that depends only on the position $F(x)$.

For this type of force, it is always possible to define a differentiable function $U(x)$ in such a way that

$$F = -\frac{dU}{dx} \tag{2.26}$$

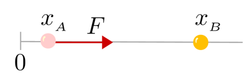


Fig. 2.11: One dimensional case. Work done by a force $\vec{F}$ from x_A to x_B by along the only possible path

This is one example of conservative force. The potential energy is defined as

$$U(x) = U(0) - \int_0^x F\,dx \tag{2.27}$$

The work done by the conservative force $F(x)$ when the particle moves between the initial x_A and final x_B is

$$W = \int_{x_A}^{x_B} F(x)dx = -\int_{x_A}^{x_B} \frac{dU(x)}{dx}dx = -\int_{x_A}^{x_B} dU(x) = U(x_A) - U(x_B) \tag{2.28}$$

The potential energy and conservative concepts can be generalized if we apply the appropriate definitions.

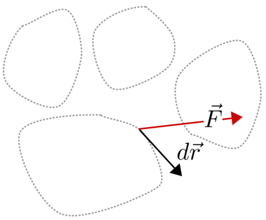


Fig. 2.12: Possible closed paths

➔ **Conservative force**. One force $\vec{F}(\vec{r})$ is conservative if, for all closed paths, the work done is null (see Figure 2.12):

$$\oint \vec{F} \cdot d\vec{r} = 0 \tag{2.29}$$

If $\vec{F}(\vec{r})$ is conservative, the work done for going from one point to another does not depend on the specific path we use, as we can see by observing Figure 2.13:

$$\oint_{C_1 \cup C_2} \vec{F} \cdot d\vec{r} = \int_{P_1:C_1}^{P_2} \vec{F} \cdot d\vec{r} + \int_{P_2:C_2}^{P_1} \vec{F} \cdot d\vec{r} = \int_{P_1:C_1}^{P_2} \vec{F} \cdot d\vec{r} - \int_{P_1:C_2}^{P_2} \vec{F} \cdot d\vec{r} \tag{2.30}$$

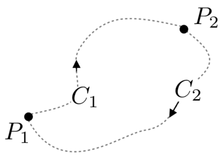


Fig. 2.13: We can go and come back to P_1 by a closed path that passes through P_2, which we can interpret as formed by two different paths, C_1 and C_2, going from P_1 to P_2

and if the force is conservative, we have

$$\int_{P_1:C_1}^{P_2} \vec{F} \cdot d\vec{r} = \int_{P_1:C_2}^{P_2} \vec{F} \cdot d\vec{r} \tag{2.31}$$

As it does not depend on the path, the work integral can be expressed as a function that is dependent only on the departure and arrival points P_1 and P_2:

$$\int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = f(P_1, P_2) \tag{2.32}$$

Now, considering the general properties of defined integrals, we have:

$$W = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{r} = \int_{P_1}^{P_0} \vec{F}\cdot d\vec{r} + \int_{P_0}^{P_2} \vec{F}\cdot d\vec{r} \tag{2.33}$$

$$= \int_{P_1}^{P_0} \vec{F}\cdot d\vec{r} - \int_{P_2}^{P_0} \vec{F}\cdot d\vec{r} = f(P_1, P_0) - f(P_2, P_0) \tag{2.34}$$

➔ **Potential energy of a conservative force**. The potential energy associated with a conservative force can be defined as the point P dependent function, $U(P)$, in accordance with

$$U(P) - U(P_0) = -\int_{P_0}^{P} \vec{F}\cdot d\vec{r} \tag{2.35}$$

The integral depends only on $P = (x, y, z)$ and $P_0 = (x_0, y_0, z_0)$, with the latter point being a reference point. Thus, the value of the potential energy $U(x, y, z)$ is expressed in terms of a reference value $U(x_0, y_0, z_0)$ for a certain position (x_0, y_0, z_0). If it suits us, we can always adjust the potential energy function U so that the value at the reference point is zero, $U(P_0) = U(x_0, y_0, z_0) = 0$. For example, in the case of a particle attached to a spring, the potential energy can be taken as zero when the spring is not deformed (neither stretched nor compressed). In the case of the gravitational force exerted by two particles of a given mass, the potential energy can be taken as zero when the distance between the two particles is infinite. Taking one or another reference point only introduces one constant in the definition of potential energy. We will see that this constant has no relevance and, in practice, what matters is the *potential energy function* as a primitive of the integral $U(\vec{r}) = -\int \vec{F}\cdot d\vec{r}$.

From the definition of potential energy, we have

$$U(P) = -\int_{0}^{P} \vec{F}\cdot d\vec{r} = \int_{P}^{0} \vec{F}\cdot d\vec{r} = f(P, 0) \tag{2.36}$$

Thus, combining (2.33) and (2.36), we have

$$W = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{r} = U(P_1) - U(P_2) = -\int_{P_1}^{P_2} dU \tag{2.37}$$

➔ **Work done by a conservative force**. The work of a conservative force is equal to the decrease in its potential energy

$$W = -\Delta U \tag{2.38}$$

If we know the potential energy U associated with a force $\vec{F}$ but we do not know the force, we can find it by taking into account that $\vec{F} \cdot d\vec{r} = -dU = -\vec{\nabla} U \cdot d\vec{r}$ and, therefore

$$\vec{F} = -\vec{\nabla} U \tag{2.39}$$

Generally, it is more practical to use the definition of potential energy as a primitive integral instead of using the integration limits and adding an indeterminate constant, which we can choose at will:

$$U(\vec{r}) = -\int \vec{F} \cdot d\vec{r} + \text{constant} \tag{2.40}$$

We do not have to worry about knowing this constant as what is important is the work or the force, which depend on the potential *differences*. Only the potential differences will have a physical meaning because they are the ones that are directly related to work or force.

To know if a given force is conservative or not, it is totally unfeasible to determine this by means of the literal definition of conservative force because we would need to test all conceivable closed paths. If we find a closed path in which the work is different from zero, we can say it is not conservative; but if we do not find it, we will have to keep looking! In some cases, a very simple criterion is that a force is conservative if it is not necessary to specify a detailed path when carrying out the integral $\int \vec{F} \cdot d\vec{r}$. In general, we can write

$$\int \vec{F} \cdot d\vec{r} = \int F_x dx + \int F_y dy + \int F_z dz$$

If F_x depends only on x, F_y depends only on y and F_z depends only on z, then all three integrals will always be feasible and the force will be conservative. If, for example, F_y depends on x, we will need to know a relationship between the x and y coordinates in order to be able to integrate them, and this means choosing a path. This criterion can also be used in cases of high symmetry, as in the case of gravitational and electrostatic forces (both of which have spherical symmetry) which we will see in Chapter 3.

In the general case, there are techniques to know if a given force is conservative or not. But both this application and its proof are outside the scope of this course.

Mechanical energy

➔ **Mechanical energy**. With conservative forces, the mechanical energy is defined as the sum of the kinetic and potential energies:

$$E = E_c + U \tag{2.41}$$

If all forces are conservative, we apply the result of the work-energy theorem using the initial and final states, which we will denominate, respectively, as ini and fi. Their kinetic and potential energies are thus $E_{c:\text{ini}}$, U_{ini}, and $E_{c:\text{fi}}$, U_{fi} , giving us

$$W = E_{c:\text{fi}} - E_{c:\text{ini}} = -\left(U_{\text{fi}} - U_{\text{ini}}\right) \tag{2.42}$$

from which we obtain $E_{c:\text{fi}} + U_{\text{fi}} = E_{c:\text{ini}} + U_{\text{ini}}$ or more synthetically:

$$E_{\text{fi}} = E_{\text{ini}} \tag{2.43}$$

➔ ***Conservation of mechanical energy theorem***: *If a particle is subjected only to conservative forces, the numerical value of the mechanical energy remains constant over time.*

It is usually said that *mechanical energy is conserved*.

We can deduce the energy conservation by taking the time derivative of the energy function $E(\vec{r}, \vec{v}) = \frac{1}{2}m\vec{v}^2 + U(\vec{r})$, then take into account the following.

1) $\vec{F} \cdot d\vec{r} = -dU$ allows writing $\frac{dU}{dt} = -\vec{F} \cdot \vec{v}$.

2) $\frac{d\vec{v}^2}{dt} = 2\vec{v} \cdot \frac{d\vec{v}}{dt}$.

3) It is a total derivative, i.e. a derivative over the time path fulfilling $m\vec{a} = \vec{F}$.

By using these three results, we obtain:

$$\frac{dE}{dt} = \frac{1}{2}m\frac{d\vec{v}^2}{dt} + \frac{dU}{dt} = m\vec{v} \cdot \vec{a} - \vec{F} \cdot \vec{v} = \left(m \cdot \vec{a} - \vec{F}\right) \cdot \vec{v} = 0 \tag{2.44}$$

If the particle is subjected to several forces, some of which are conservative and others not, we can group them together. $\vec{F}$ will be the net force of the conservative forces and $\vec{F}_{NC}$ will be the net force of the non-conservative forces. On the one hand, the work of all the forces is $W = W_C + W_{NC}$ and, on the other, the equation of motion is $m\vec{a} = \vec{F} + \vec{F}_{NC}$. Only $\vec{F}$ has an associated potential energy: $W_C = \int \vec{F} \cdot d\vec{r} = -\Delta U$.

Now the energy will not be conserved, but we can write a modification of (2.43) as

$$E_{\text{fi}} - E_{\text{ini}} = W_{NC} \tag{2.45}$$

and another modification of (2.44) as

$$\frac{dE}{dt} = \vec{F}_{NC} \cdot \vec{v} \tag{2.46}$$

➔ **The unit of work and energy** in SI units is the joule (J): $1\,\text{J} = 1\,\text{N}\,\text{m}$.

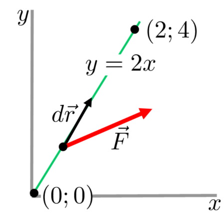
➔ **The unit of power** in SI units is the watt (W): $1\,\text{W} = 1\,\frac{\text{J}}{\text{s}}$.

Problem 2.4.1. A particle is subjected to the force $\vec{F} = (y^2 - x^2)\,\hat{\imath} + 3xy\,\hat{\jmath}$ (SI units). Find the work done by $\vec{F}$ when the particle moves from point $(0,0)$ to point $(2,4)$ following:

a) the straight line $y = 2x$

b) the parabola $y = x^2$

c) Is it conservative?

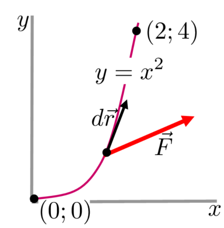


Solution figure of **a** from 2.4.1

Solution

a) The straight line $y = 2x$. In general, to calculate the work we need to know the geometric trajectory in the parametric form, $\vec{r}(\lambda)$, as well as the force $\vec{F}$ and the displacement $d\vec{r}$ on this trajectory. The parametric form of the $y = 2x$ straight line can be expressed by using the λ parameter as $\vec{r}(\lambda) = (\lambda, 2\lambda)$, with $\lambda \in [0,2]$. By substituting in the force, we obtain $\vec{F} = \left(((2\lambda)^2 - \lambda^2), 3\lambda \cdot 2\lambda\right) = 3\lambda^2\,(1,2)$; and if we differentiate $\vec{r}(\lambda)$, we obtain $d\vec{r} = (1,2)\,d\lambda$.

$$W = \int_A^B \vec{F}\cdot d\vec{r} = \int_0^2 3\lambda^2\,(1,2)\cdot(1,2)\,d\lambda = 15\int_0^2 \lambda^2 d\lambda = \boxed{40\text{ J}}$$



Solution figure of **b** from 2.4.1

b) The parabola $y = x^2$. In this case the parametric form can be written as $\vec{r}(\lambda) = (\lambda, \lambda^2)$ with $\lambda \in [0,2]$. The force on the trajectory is $\vec{F} = \left(((\lambda^2)^2 - \lambda^2), 3\lambda\cdot\lambda^2\right) = (\lambda^4 - \lambda^2, 3\lambda^3)$, and the displacement on the trajectory is $d\vec{r} = (1.2\lambda)\,d\lambda$. We obtain

$$W = \int_A^B \vec{F}\cdot d\vec{r} = \int_0^2 (\lambda^4 - \lambda^2, 3\lambda^3)\cdot(1,2\lambda)\,d\lambda = \int_0^2 (7\lambda^4 - \lambda^2)\,d\lambda = \boxed{42.13\text{ J}}$$

It is not conservative because there is at least one case in which the work between two points depends on the path. Note that if the work had been the same, we could not guarantee it is conservative. ■

Problem 2.4.2. With an adequate initial velocity and due to the force $\vec{F} = -4(x,y)$, one particle of mass 4 moves from point $(2,0)$ to point $(0,3)$ through the trajectory $\vec{r}(t) = (2\cos t, 3\sin t)$ (all magnitudes are expressed in SI units):

a) Prove that this trajectory is possible.

b) Calculate the work of the force between these two points through an arbitrary path.

c) Is this force conservative? Find the associated potential energy.

d) Calculate the mechanical energy and explicitly prove that it is conserved along the given trajectory.

Solution

a) We have to prove that the trajectory satisfies Newton's law of motion for this force: $\vec{F} \stackrel{?}{=} m\ddot{\vec{r}}$.

The force along the trajectory is

$$\vec{F} = -4(x, y) = -4\vec{r} = -4(2\cos t, 3\sin t)$$

The acceleration is

$$\dot{\vec{r}} = (-2\sin t, 3\cos t)$$
$$\ddot{\vec{r}} = (-2\cos t, -3\sin t)$$

Thus

$$m\,\ddot{\vec{r}} = -4(2\cos t, 3\sin t)$$

which is the same as $\vec{F}$ and, therefore, the law of motion is satisfied.

b) and **c)**: For the structure of force $\vec{F}$, of type $\vec{F} = (F_x(x), F_y(y), F_z(z))$, we see that it is conservative. The potential energy is

$$U = -\int \vec{F} \cdot d\vec{r} = 4\int (xdx + ydy) = 4\left(\int xdx + ydy\right) = 2(x^2 + y^2) + C$$

The work is

$$W = -\Delta U = -(U(0,3) - U(2,0)) = -10\text{ J}$$

d) The energy of the system is

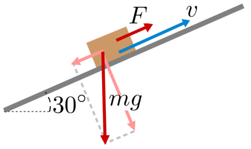
$$E = E_c + U = \frac{1}{2}4\,v^2 + 2(x^2 + y^2) + C$$

By substituting the trajectory, taking into account that $v^2 = \dot{x}^2 + \dot{y}^2$ and simplifying, we obtain

$$E = \frac{1}{2}4(4\sin^2 t + 9\cos^2 t) + 2(4\cos^2 t + 9\sin^2 t) + C = 26 + C$$

E remains constant. ■

Problem 2.4.3. A mass of 5 kg rises by a plane inclined at 30° to the horizontal at a constant speed of 15 m/s. Calculate the power of the force that makes it rise.



Solution to Problem 2.4.3

Solution

If the force that makes it rise is $\vec{F}$ and the velocity is $\vec{v}$, the power will be $P = \frac{dW}{dt} = \vec{F} \cdot \vec{v}$.

The force that makes the mass rise must counteract the component of the weight parallel to the inclined plane and, since the velocity is constant, the total force is null. Thus, the mentioned force has the same modulus as the weight component: $F = mg\sin 30°$. As this force has the same direction as the velocity, we have

$$P = \vec{F} \cdot \vec{v} = Fv = mgv\sin 30° = 367.5\text{ W}$$

■

Problem 2.4.4. Find the expression of the potential energy associated with the force $\vec{F} = \left(-kx + \frac{a}{x^3}\right)\hat{\imath}$.

Solution

$$U = -\int \vec{F}\cdot d\vec{r} = -\int F\,dx = -\int \left(-kx + a/x^3\right) dx = \frac{1}{2}kx^2 + \frac{a}{2x^2} + C \quad \blacksquare$$

2.5 Some forces

As useful examples, we will apply the discussed concepts to some kinds of forces acting on a particle. These will appear later in different contexts, but dealing them here in the same block allows us to see how useful the concepts in this chapter are, appreciate what they have in common and see how they differ.

Null Force

$\vec{F} = 0$. The equation of motion is

$$m\,\vec{a} = 0 \Rightarrow \frac{d^2\vec{r}}{dt^2} = 0 \tag{2.47}$$

By integrating it twice we get $\vec{r}(t) = \vec{C}_1 + \vec{C}_2\,t$. By taking into account the initial conditions $\vec{r}(t_0) = \vec{r}_0$ and $\dot{\vec{r}}(t_0) = \vec{v}_0$:

$$\left.\begin{array}{l} \vec{r}(t_0) = \vec{r}_0 = \vec{C}_1 + \vec{C}_2\,t_0 \\ \dot{\vec{r}}(t_0) = \vec{v}_0 = \vec{C}_2 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \vec{C}_2 = \vec{v}_0 \\ \vec{C}_1 = \vec{r}_0 - \vec{v}_0\,t_0 \end{array}\right.$$

We finally obtain

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0\,(t - t_0) \tag{2.48}$$

The potential energy function is $dU = -\vec{F}\cdot d\vec{r} = 0 \Rightarrow U = \text{ct}$. The energy is conserved and it will only be kinetic energy. Momentum and angular momentum are also conserved.

Constant Force

Constant force means $\vec{F} = \text{constant}$. The motion is contained in the plane defined by the initial velocity and the force (see Figure 2.14), since, according to Newton's second law of motion, changes in the velocity vector's direction have the direction of the force vector: $d\vec{v} = \frac{\vec{F}}{m}dt$.

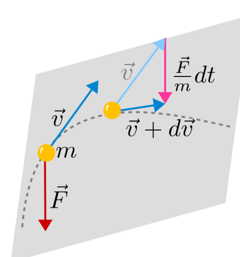


Fig. 2.14: Constant force in an unspecified direction

The equation of motion is

$$m\,\vec{a} = \vec{F} \Rightarrow \frac{d^2\vec{r}}{dt^2} = \frac{\vec{F}}{m} = \text{ct} \tag{2.49}$$

By integrating it twice we get $\vec{r}(t) = \vec{C}_1 + \vec{C}_2\, t + \frac{1}{2}\frac{\vec{F}}{m}t^2$ and considering the initial conditions $\vec{r}(t_0) = \vec{r}_0$ and $\dot{\vec{r}}(t_0) = \vec{v}_0$

$$\left.\begin{array}{l} \vec{r}(t_0) = \vec{r}_0 = \vec{C}_1 + \vec{C}_2\, t_0 + \frac{1}{2}\frac{\vec{F}}{m}t_0^2 \\ \dot{\vec{r}}(t_0) = \vec{v}_0 = \vec{C}_2 + \frac{\vec{F}}{m}t_0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \vec{C}_2 = \vec{v}_0 - \frac{\vec{F}}{m}t_0 \\ \vec{C}_1 = \vec{r}_0 - \vec{v}_0 t_0 + \frac{1}{2}\frac{\vec{F}}{m}t_0^2 \end{array}\right.$$

we get the trajectory expression

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0\,(t - t_0) + \frac{1}{2}\frac{\vec{F}}{m}(t - t_0)^2 \tag{2.50}$$

If we choose the reference frame so that the plane of motion is (x, y) and the y-axis in the direction of the force $\vec{F} = (0, F)$ (see Figure 2.15)

$$\left\{\begin{array}{l} x(t) = x_0 + v_{x0}\,(t - t_0) \\ y(t) = y_0 + v_{y0}\,(t - t_0) + \frac{1}{2}\frac{F}{m}(t - t_0)^2 \end{array}\right.$$

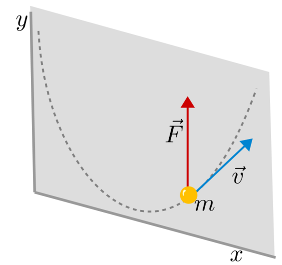


Fig. 2.15: Constant force in the direction of increasing y

$\vec{F} = $ ct is a conservative force, as we can find the associated potential energy:

$$U = -\int \vec{F}\cdot\, d\vec{r} = -\vec{F}\cdot\vec{r} + \text{ct} \tag{2.51}$$

An important case (but not the only one!) is that of a particle of mass m, subjected to the gravitational field $\vec{g}$ =constant, close to the Earth or some other planet (see Figure 2.16):

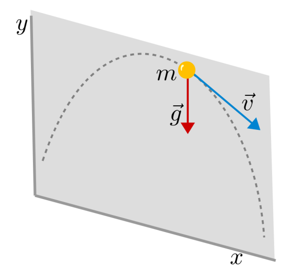


Fig. 2.16: Constant gravitational field g in the direction of decreasing y

$$m\,\vec{a} = \vec{F} = m\vec{g} \Rightarrow \vec{a} = \frac{\vec{F}}{m} = \vec{g} = \text{ct} \tag{2.52}$$

With the above-mentioned choice of the axis and choosing the positive direction of the y axis in the opposite direction of the force $\vec{g} = (0, -g)$, it results in

$$\left\{\begin{array}{l} x(t) = x_0 + v_{x0}\,(t - t_0) \\ y(t) = y_0 + v_{y0}\,(t - t_0) - \frac{1}{2}g(t - t_0)^2 \end{array}\right.$$

and the potential energy, by using (2.51), is $U = m\,g\,y + $ ct.

Another case of interest is that of a particle of mass m and electric charge q under the action of a constant electric field $\vec{E} = $ ct (see Figure 2.17):

$$m\,\vec{a} = \vec{F} = q\,\vec{E} \Rightarrow \vec{a} = \frac{q\,\vec{E}}{m} = \text{ct}$$

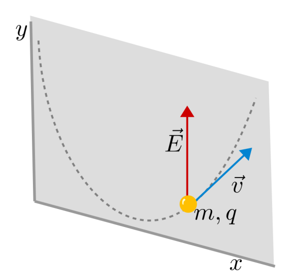


Fig. 2.17: Constant electric field E in the direction of increasing y

With the above-mentioned choice of the axis and by choosing the positive direction of the y axis in the direction of the field $\vec{E}$, we have

$$\left\{\begin{array}{l} x(t) = x_0 + v_{x0}\,(t - t_0) \\ y(t) = y_0 + v_{y0}\,(t - t_0) + \frac{1}{2}\frac{q\,E}{m}(t - t_0)^2 \end{array}\right.$$

And by using (2.51), the resulting potential energy is $U = -q\,E\,y + $ ct.

$\vec{F}(\vec{r})$ unidimensional motion

If the force has a fixed direction, with a suitable choice of axes we can express it as $\vec{F} = F(\vec{r})\,\hat{\imath}$. If the initial velocity has the same direction as the force, the force will not change the direction of velocity and the motion can be described with a single x coordinate (see Figure 2.18):

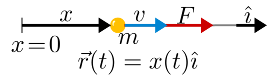


Fig. 2.18: One-dimensional motion

$$m\,\ddot{x} = F(x)$$

The force will always be conservative, since

$$U(x) = -\int \vec{F}\,(\vec{r})\,d\vec{r} = -\int F\,(x)\,dx$$

and the integral can always be calculated.

Because mechanical energy is conserved, we can take advantage of this fact to find the trajectory. E =constant and also

$$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + U(x) = \text{ct}$$

which is a differential equation for finding the trajectory $x(t)$. We can separate the variables x and t and integrate t between the limits t_0 to t, and x between the limits $x(t_0)$ to $x(t)$. The integral in t is immediate:

$$\int_{x(t_0)}^{x(t)} \frac{dx}{\sqrt{E - U(x)}} = \sqrt{\frac{2}{m}}(t - t_0) \tag{2.53}$$

If we know explicitly the function $U(x)$, we can always perform the integral in (2.53) and then isolate $x(t)$. The two constants that appear in the trajectory are E and $x_0 = x(t_0)$. Although the problem is formally solved, it must be said that the remaining integral can be very complicated.

One-dimensional harmonic motion

This is a special case from the previous subsection

$$F = -k\,x + F_0$$

It is usually written as $F = -k(x - x_0)$, that is to say, $x = x_0$ is the position for which the force is null: $x_0 = \frac{F_0}{k}$ (see Figure 2.19).

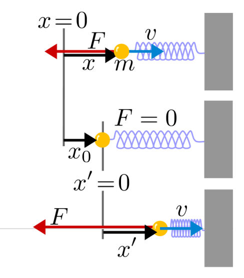


Fig. 2.19: One-dimensional harmonic motion

Having this provides a new reference frame, x', such that $x' = x - x_0$. The force will be expressed as $F = -k\,x'$. For convenience, we rename the x' coordinate as x. In fact, the force we want to deal with is

$$F = -k\,x$$

The equation of motion is

$$m\,\ddot{x} = -k\,x$$

The potential energy is

$$U(x) = -\int(-kx)\,dx = \frac{1}{2}k\,x^2 + \text{ct}$$

And we can find the trajectory through the integral (2.53), which now is

$$\int_{x(t_0)}^{x(t)} \frac{dx}{\sqrt{E - \frac{1}{2}k\,x^2}} = \sqrt{\frac{2}{m}}(t - t_0)$$

By taking the common factor $\frac{k}{2}$ in the root of the denominator (for this we multiply and divide E by $\frac{k}{2}$), the first member can be written as

$$\sqrt{\frac{2}{k}} \int_{x(t_0)}^{x(t)} \frac{dx}{\sqrt{\frac{2E}{k} - x^2}}$$

Taking into into account that

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right)$$

and once the integration limits have been replaced and matching the second member, we obtain

$$\sqrt{\frac{2}{k}}\left\{\arcsin\left(\sqrt{\frac{k}{2E}}x\right) - \arcsin\left(\sqrt{\frac{k}{2E}}x_0\right)\right\} = \sqrt{\frac{2}{m}}(t - t_0)$$

Now, isolating x, we find the trajectory

$$x(t) = A\sin\left(\omega\,(t - t_0) + \varphi_0\right) \tag{2.54}$$

with $A = \sqrt{\frac{2E}{k}}$, $\sin\varphi_0 = \frac{x_0}{A}$, $\omega = \sqrt{\frac{k}{m}}$. It mus be remembered that we have the relationship $E = \frac{1}{2}m\,v_0^2(t_0)^2 + \frac{1}{2}k\,x_0^2$. Note that the relationship between A and E can also be written as $E = \frac{1}{2}kA^2$.

Harmonic motions will be studied in depth in Chapter 6.

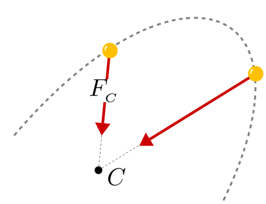


Fig. 2.20: Central force

Central forces

Central forces have a line of action that always passes through the same point (see Figure 2.20). Some examples are the gravitational force that the Sun (or any other mass) exerts on a particle; electrostatic force of a fixed charge on a charged particle; and the tension of a rope passing through a small pulley with one end tied to a body (see Figure 2.21). In Figure 2.21, we see that tension T_B can be a constant force but tension T is a central force!

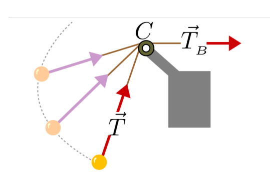


Fig. 2.21: Central force caused by the tension in a rope passing through a small pulley

All the central forces fulfil Kepler's law.

Proof. The torque $\vec{M}_{(C)}$ of the force with respect to point C is always null. Taking into account that $\frac{d\vec{L}_{(C)}}{dt} = \vec{M}_{(C)}$ and if no other force acts, we deduce $\frac{d\vec{L}_{(C)}}{dt} = 0$. The angular momentum with respect to C is conserved throughout the motion and, as we have seen in Section 2.3, this is equivalent to Kepler's second law. ■

Potential energy of a central force of the type $\vec{F} = -F(r)\,\hat{r}$ (see Figure 2.22)

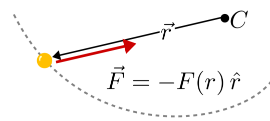


Fig. 2.22: Vector expression of a central force

Let us try to calculate the work integral regardless of the path:

$$\int \vec{F} \cdot d\vec{r} = -\int F\,\hat{r} \cdot d\vec{r}$$

Now

$$\left.\begin{array}{l} d(\vec{r}\cdot\vec{r}) = 2\vec{r}\cdot d\vec{r} \\ dr^2 = 2r dr \end{array}\right\} \Rightarrow \vec{r}\cdot d\vec{r} = r dr \Rightarrow \hat{r}\cdot d\vec{r} = dr$$

Therefore, the integral involves a single variable r and we have not chosen any particular path. Central forces of type $\vec{F} = -F(r)\,\hat{r}$ are conservative with an associated potential energy:

$$U = \int F(r)\, dr + \text{ct} \tag{2.55}$$

For example, the force exerted by the Sun (fixed, of mass M) on a planet (particle of mass m) is

$$\vec{F} = -G\frac{mM}{r^2}\,\hat{r}$$

Substituting this force in (2.55) gives the potential energy:

$$U = -G\frac{mM}{r} + \text{ct}$$

Non-conservative forces

A force is called ***gyroscopic*** when the potential energy function cannot be defined, but $\frac{dE}{dt} = 0$. From an energy point of view, they can be considered conservative, since they do not dissipate energy. A force is called ***dissipative*** when the potential energy function cannot be defined and $\frac{dE}{dt} < 0$.

Gyroscopic forces

The best known example of a gyroscopic force is the Lorentz force $\vec{F}_B$ due to a magnetic field $\vec{B}$ acting on a charged particle q moving at a velocity $\vec{v}$:

$$\vec{F}_B = q\,\vec{v}\times\vec{B} \tag{2.56}$$

Let us suppose that a particle is subjected to a conservative force $\vec{F}$, with potential energy $U(\vec{r})$, and to a Lorentz force (2.56). According to what we have seen, in regard to energy and non-conservative forces, (2.46), we have

$$\frac{dE}{dt} = \vec{v} \cdot \left(q\,\vec{v} \times \vec{B}\right) = 0$$

Energy E is conserved!

Dry friction forces

When a rigid object slides in contact with a non-smooth surface that we assume to be at rest, whether through gravity or due to any other force acting on it, the points of contact contain distributed forces that, if the object is small, can be decomposed into two: one perpendicular to the contact surface, which we call ***normal***, N; and another one opposite to the motion, which we call ***dry friction***, F_f. For surfaces that are sufficiently smooth and hard while the object is moving, the modulus of the dry friction force is $F_f = \mu_k N$, where μ_k is the **kinematic or dynamic friction coefficient**, which depends on the nature of the surfaces in contact. The force F_f can be expressed in the following vector form (see Figure 2.23):

$$\vec{F}_f = -F_f \frac{\vec{v}}{v}, \quad F_f = \mu_k N \tag{2.57}$$

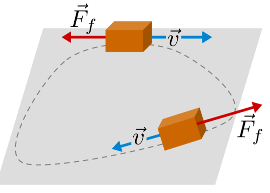


Fig. 2.23: The dynamic friction force has the opposite direction of velocity

Thus, it is clear that it is a force that depends on the velocity (not on the modulus, but on the direction!). It is only in the case that the surface causing the friction force is at rest that we can say it is *always opposed to the motion*. It is the dependence on the velocity that prevents us from finding the potential energy. When we want to calculate the work integral, we need to know the path followed by the object.

Let us suppose that the particle is subjected to a conservative force, $\vec{F}$, with potential energy $U(\vec{r})$ and to a dry friction force. According to what we have seen concerning the energy and non-conservative forces (2.46), we will have

$$\frac{dE}{dt} = \vec{v} \cdot \left(-F_f \frac{\vec{v}}{v}\right) = -F_f\, v$$

A dry friction force is dissipative. From a mechanical point of view, this energy *disappears*. A physical interpretation beyond mechanics can be found by applying conservation of energy or the first principle of thermodynamics: some of this energy increases the internal energy of the system and the rest dissipates in the form of heat/radiation. However, understanding this requires going a bit into thermodynamics which is beyond the scope of this course.

We have seen the effect of dry friction when an object moves, generally due to the fact of being subjected to another force that exceeds the friction. But what happens if the object does not move?

If a force F is applied to the block in Figure 2.24 and friction exists between the block and the surface supporting the block, the friction force evolves as follows.

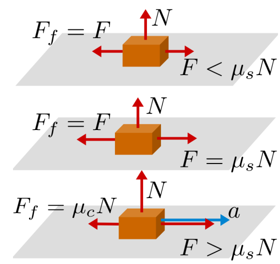


Fig. 2.24: Evolution of the friction force

If the force is not great enough, the block does not move and $F_f = F$. If we increase F, a boundary value is reached, $F = \mu_s N$. Even though the block does not move (again, $F_f = F$), any increase in F, no matter how small, will make it move. We say that it is in **imminent motion**. Coefficient μ_s is called the ***static friction coefficient***. If we continue to increase F, then $F > \mu_s N$, whereby the block moves and the friction force decreases to the value $F_f = \mu_k N$. Coefficient μ_k is, already discussed above, the kinetic or dynamic coefficient of friction, which always $\mu_s > \mu_k$. The graph in Figure 2.25 illustrates the evolution of the friction force F_f as a function of the applied force F.

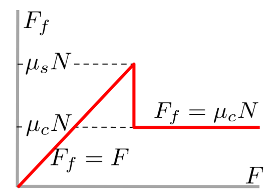


Fig. 2.25: Evolution of the friction force F_f as a function of the applied force F

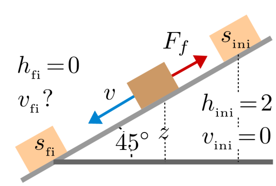
Note: Although real problems must take into account this behaviour and, therefore, the existence of both coefficients, this course will use the friction coefficient μ without any specification (unless otherwise stated) and we will understand that $\mu = \mu_s = \mu_k$.

Problem 2.5.1. A particle of 2 kg slides on an inclined plane 2 m high. Starting from the top with null speed, it reaches the end at a speed of 5 m/s. Calculate the energy loss due to the friction force and the value of this force considered to be constant, if the angle between the plane and the horizontal is 45°. What is the value of the dynamic coefficient of friction?

Solution

The loss of mechanical energy is the increase in mechanical energy with changed sign:

$$E = \tfrac{1}{2}mv^2 + mgz$$
$$-\Delta E = -\left(E_{\text{fi}} - E_{\text{ini}}\right) = E_{\text{ini}} - E_{\text{fi}} =$$
$$\left(\tfrac{1}{2}m\, v_{\text{ini}}^2 + mgh_{\text{ini}}\right) - \left(\tfrac{1}{2}m\, v_{\text{fi}}^2 + mgh_{\text{fi}}\right) = 14.2\ \text{J}$$

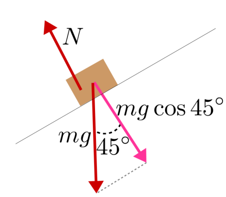


Solution to Problem 2.5.1

The energy variation is due to the work of the friction force:

$$\Delta E = W_f = \int_{\text{ini}}^{fi} \vec{F}_f \cdot d\vec{r} = -\int_{s_{\text{ini}}}^{s_{\text{fi}}} F_f\, ds = -F_f\left(s_{\text{fi}} - s_{\text{ini}}\right) = -F_f \frac{h_{\text{ini}}}{\sin 45°}$$

where $ds = |d\vec{r}|$ is the length travelled by the particle along the inclined plane and the friction force $\vec{F}_f$ always has an opposite direction to velocity.



Solution to Problem 2.5.1

Finally, the friction force is

$$F_f = -\frac{\Delta E}{\left(\frac{h_{\text{ini}}}{\sin 45°}\right)} = 5.02\,\text{N}$$

The dynamic coefficient of friction is

$$F_f = \mu N \Rightarrow \mu = \frac{F_f}{N} = \frac{F_f}{mg\cos 45^0} = 0.36$$

■

Viscous friction forces

We know through experimentation that the friction of a body in a fluid (aerodynamic or viscous friction) depends on the shape of the body, the material characteristics of the surface, the type of fluid and the velocity of the body (with respect to the fluid, which we will consider here to be at rest). A good approach to this type of friction is, in the laminar flow regime, (see upper part of Figure 2.26):

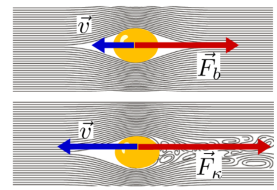


Fig. 2.26: Upper: the laminar regime occurs for high viscosities and low relative velocities. Down: the turbulent regime is due to low fluid viscosities and high relative velocities

$$\vec{F}_b = -b\vec{v} \quad b > 0 \tag{2.58}$$

It is not conservative. Let us suppose that the particle is being submitted to a conservative force $\vec{F}$, with potential energy $U(\vec{r})$, and to a viscous friction force. According to what we have seen concerning energy and non-conservative forces (2.46), we have

$$\frac{dE}{dt} = \vec{v} \cdot (-b\vec{v}) = -b\, v^2$$

If the regime is turbulent (see lower part of Figure 2.26), the friction force can be approximated as

$$\vec{F}_\kappa = -\kappa v^2\, \hat{v} \quad \kappa > 0 \tag{2.59}$$

Problem 2.5.2. A particle in a viscous medium is subjected to a constant force in addition to the frictional force due to viscosity.

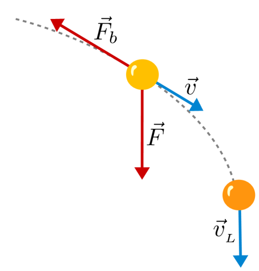
a) Prove that the particle ends up having a constant velocity and determine this velocity.

b) Apply this result to the case of it falling in a viscous medium.

Solution

a) The force acting on the particle is $\vec{F}_T = \vec{F} + \vec{F}_b = m\vec{a}$, with $\vec{F}$ being constant and $\vec{F}_b = -b\vec{v}$, with $b > 0$ being constant.

Looking at the figure, we can see that $\vec{F}_b$ tends to decrease the velocity component normal to $\vec{F}$, so that $\vec{v}$ and also $\vec{F}_b$ itself will align with $\vec{F}$. This process will end when $\vec{F}_T = 0$, that is, when the velocity reaches a value $\vec{v}_L$ such that $\vec{F} - b\vec{v}_L = 0$. We thus obtain



Solution to Problem 2.5.2

$$\vec{v}_L = \frac{\vec{F}}{b}$$

Note that this is a limit value. $\vec{v} = \frac{\vec{F}}{b}$ is one possible solution to the equation of motion that needs an initial given velocity $\vec{v}_0 = \frac{\vec{F}}{b}$.The general solution for the evolution of the velocity from an initial velocity $\vec{v}_0$ must be obtained by integrating of $m\vec{a} = \vec{F} - b\vec{v}$, which we can write as $\dot{\vec{v}} + \frac{b}{m}\vec{v} = \frac{b}{m}\vec{v}_L$. The solution to this equation is

$$\vec{v}(t) = (\vec{v}_0 - \vec{v}_L)\, \mathrm{e}^{-\dfrac{b}{m}t} + \vec{v}_L$$

We can see that for $t \to \infty$, we have $\vec{v} \to \vec{v}_L$.

b)

$$\vec{F} = m\vec{g} \Rightarrow \boxed{\vec{v}_L = \frac{m\vec{g}}{b}} \qquad \blacksquare$$

Forces that only depend on time

These are forces in the form of $\vec{F}_t = \vec{F}_t(t)$. They are not conservative because they explicitly depend on time. Let us suppose that the particle is subjected to a conservative force, $\vec{F}$, with potential energy $U(\vec{r})$, and to a force $\vec{F}_t = \vec{F}_t(t)$. According to what we have seen concerning energy and non-conservative forces (2.46),we have

$$\frac{dE}{dt} = \vec{v} \cdot \vec{F}_t$$

If the particle is subjected only to $\vec{F}_t$, the formal integration of the equation of motion is simple:

$$m\,\frac{d^2\vec{r}}{dt^2} = \vec{F}_t(t) \Rightarrow \vec{r}(t) = \frac{1}{m}\int\left(\int \vec{F}_t(t)\,dt\right)dt \tag{2.60}$$

The initial conditions $\vec{r}(t_0) = \vec{r}_0$ and $\dot{\vec{r}}(t_0) = \vec{v}_0$ determine the integration constants.

Problem 2.5.3. A particle of mass m at rest at $\vec{r} = (0,0,0)$ is subjected to the force depending on time $\vec{F} = \vec{F}_0 \sin\Omega t$.

a) Find the time trajectory.

b) What is the initial velocity that makes the motion the same as that of a spring?

Solution

a) We find the velocity:

$$\vec{v} = \frac{1}{m}\int \vec{F}dt = \frac{1}{m}\int \vec{F}_0 \sin\Omega t\; dt = -\frac{\vec{F}_0}{m\Omega}\cos\Omega t + C_1$$

We find the position:

$$\vec{r} = \int \vec{v}dt = -\frac{\vec{F}_0}{m\Omega^2}\sin\Omega t + \vec{C}_1 t + \vec{C}_2$$

We find the integrations constants:

$$\vec{r}(0) = (0,0,0) \Rightarrow \vec{C}_2 = (0,0,0)$$
$$\vec{v}(0) = (0,0,0) \Rightarrow \vec{C}_1 = \tfrac{\vec{F}_0}{m\Omega}$$

We obtain:

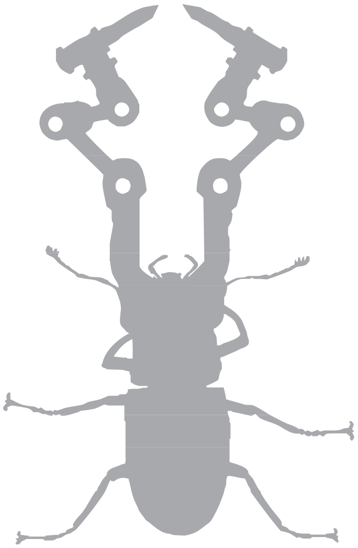
$$\boxed{\vec{r}(t) = \frac{\vec{F}_0}{m\Omega}\left(t - \frac{1}{\Omega}\sin\Omega t\right)}$$

b) For a spring

$$\vec{r} = \vec{A}\sin(\omega t + \varphi_0) \Rightarrow \ddot{\vec{r}} = -\vec{A}\omega^2 \sin(\omega t + \varphi_0) \Rightarrow \vec{F} = -m\vec{A}\omega^2 \sin(\omega t + \varphi_0)$$

If we compare this force with the one given in the statement, we need that $\varphi_0 = \pi$, $\omega = \Omega$ and $\vec{A} = \frac{\vec{F}_0}{m\Omega^2}$. Thus

$$\vec{r} = \frac{\vec{F}_0}{m\Omega^2}\sin(\Omega t + \pi) \Rightarrow \dot{\vec{r}} = \frac{\vec{F}_0}{m\Omega}\cos(\Omega t + \pi) \Rightarrow \boxed{\vec{v}(0) = -\frac{\vec{F}_0}{m\Omega}} \quad \blacksquare$$



→ 3

3 Dynamics of N particles

Introduction

We will extend the laws and concepts introduced for a particle to N particles. We will also take a look at constrained particle systems, giving special attention to the case of a rigid body. Furthermore, we will consider the case of a finite system of N discrete particles of positions $\vec{r}_i$ and masses m_i (see Figure 3.1), although we can also include continuous systems with a number of particles $N \to \infty$ with positions $\vec{r}$ and masses $dm = \rho\, dv$, where ρ is de density (see Figure 3.2). For this, we need to understand *only* the sums, with $N \to \infty$, $m_i \to dm = \rho dv$ and $\vec{r}_i \to \vec{r}$, as integrals:

$$\sum_{i=1}^{N} f(\vec{r}_i)\; m_i \to \int_V f(\vec{r})\, \rho dv \tag{3.1}$$

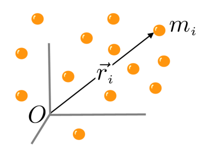


Fig. 3.1: System of N particles

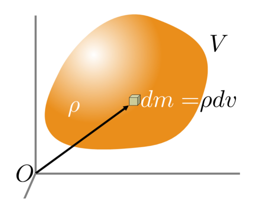


Fig. 3.2: A continuous body as a system of $N \to \infty$ particles

3.1 Forces between particles. Newton's second and third laws

Different types of forces can act on each particle in a system of N particles. According to their origin, they can be classified into two separate sets: external forces and internal forces.

➔ **External forces**. These are caused by physical agents that do not belong to the system. We will denote as $\vec{F}_i$ the net force, which is the resultant of all the external forces acting on particle i (see Figure 3.3).

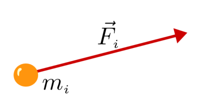


Fig. 3.3: The cause of the external force $\vec{F}_i$ is outside the system

➔ **Internal forces**. These are forces that the particles in the system exert on each other. $\vec{F}_{ij}$ is the force that particle i exerts on particle j (3.4).

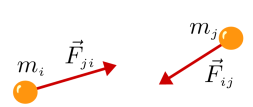


Fig. 3.4: The cause of the internal force $\vec{F}_{ij}$ is the interaction between the particles

Because we are dealing with particles, $\vec{F}_{ii} = 0$. The motion equations for a system with N particles are obtained from applying Newton's second law to each particle.

They are, therefore, N vector equations.

➔ ***Newton's second law, or the law of motion for N particles***:

$$\vec{F}_i + \sum_{j=1}^{N} \vec{F}_{ji} = m_i \vec{a}_i \tag{3.2}$$

Solving these equations is very difficult. Fortunately, we will not always be interested in finding the trajectory of each particles in a system. Either way, we will need to find general methods that allow us to obtain interesting features of the system's behaviour. One thing that greatly simplifies these equations is the fact that internal forces always fulfil the condition of action and reaction, a condition that Newton elevated to the category of law.

➔ ***Newton's third law, or the principle of action and reaction***. *The force exerted by particle j on particle i has the same modulus and opposite direction as the force acting on particle j due to particle i:* $\vec{F}_{ji} = -\vec{F}_{ij}$

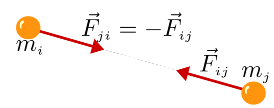


Fig. 3.5: Action-reaction law between two particles, the m_i and m_j particles

Let us note that, in order for this law to be obeyed, it is necessary that the interaction occurs instantaneously. If this were not the case, we could move particle i and, for a few moments, particle j would not know it, which would violate the third law. In relativity, this law is not fulfilled!

Although not originally made explicit in Newton's formulation, it is understood that the line of action of the two forces is the same. We will also consider it in this way (see Figure 3.5).

3.2 Net force and centre of mass

We take all the equations of motion for each particle (3.2) and add them up. The left part is the sum of all forces, regardless of where they are applied and this sum is called the ***net force*** $\vec{F}$. As a result of the action-reaction principle, $\vec{F}_{ij} + \vec{F}_{ji} = 0$; therefore, the sum of the internal forces is null. Thus, by summing all the forces of the system to find the net force $\vec{F}$ only the external ones will survive. The net force $\vec{F}$ is the sum of all external forces.

$$\vec{F} = \sum_{i=1}^{N} \left(\vec{F}_i + \sum_{j=1}^{N} \vec{F}_{ji} \right) = \sum_{i=1}^{N} \vec{F}_i \tag{3.3}$$

If the net force $\vec{F}$ were applied to a hypothetical particle whose mass would be the sum of all masses of all the particles in a given system, $m = \sum_{i=1}^{N} m_i$, the acceleration of this particle due to the force $\vec{F}$ would be $\vec{a}_{CM}$, according to

$$\vec{F} = m\,\vec{a}_{CM} \tag{3.4}$$

The point where this hypothetical particle is located is called the system's ***centre of mass*** and it is indicated by CM (see Figure 3.6). We will indicate the CM position vector with $\vec{r}_{CM}$. If $\vec{F} = 0$, CM is moving at constant velocity, $\vec{a}_{CM} = 0$. In this case, CM can be used as the origin of the inertial reference frame: is the **centre of mass reference frame** (CRF).

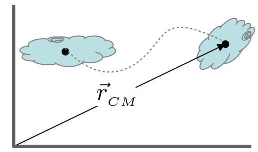


Fig. 3.6: The CM is a good representation of the overall motion of a cloud

The CM point represents an approximate and global position of the system. For example, let us think of the system as a cloud. This cloud generally moves from one place to another while its shape changes. In many cases, we are not interested in studying the change in shape but the global motion, which is represented very well by the trajectory described by the centre of mass $\vec{r}_{CM}$.

If we know the net force as a function of the position and the velocity of the CM (3.4) is the equation of motion for the CM, then knowing the position and initial velocity, allows us to obtain the trajectory $\vec{r}_{CM}(t)$. One problem that must be solved is finding the initial position and initial velocity of the centre of mass while knowing the positions and velocities of all the particles at this instant.

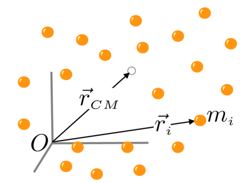


Fig. 3.7: The mass m_i and position $\vec{r}_i$ of the N particles determine $\vec{r}_{CM}$

For each instant, the position of the CM, $\vec{r}_{CM}$, can be found if we know the positions and masses of all particles at this instant, $\vec{r}_i$, m_i (see Figure 3.7):

$$\vec{r}_{CM} = \frac{1}{m}\sum_{i=1}^{N} m_i \vec{r}_i \tag{3.5}$$

Proof. Taking into account the sum of the motion equations for the N particles (3.2), and the definition of CM (3.4), we have

$$\vec{F} = \sum_{i=1}^{N} m_i \vec{a}_i = m\,\vec{a}_{CM} \Rightarrow \vec{a}_{CM} = \frac{1}{m}\sum_{i=1}^{N} m_i \vec{a}_i$$

which can be written as

$$\frac{d^2}{dt^2}\left(\vec{r}_{CM} - \frac{1}{m}\sum_{i=1}^{N} m_i \vec{r}_i\right) = 0$$

Then we integrate it, thus giving

$$\vec{r}_{CM} = \frac{1}{m}\sum_{i=1}^{N} m_i \vec{r}_i + \vec{C}_1 + \vec{C}_2\, t$$

where $\vec{C}_1$ and $\vec{C}_2$ are the two integration constants.

Now, if we have a single particle, $N = 1$, we want the CM to be wherever this particle is at all times, that is $\vec{r}_{CM} = \vec{r}_1, \Rightarrow \vec{r}_1 = \vec{r}_1 + \vec{C}_1 + \vec{C}_2\, t \Rightarrow \vec{C}_1 = \vec{C}_2 = 0$ and the expression of $\vec{r}_{CM}$ matches (3.5), which is what we wanted to prove. ■

If the system is a **continuous body** (see Figure 3.8), we will have

$$\vec{r}_{CM} = \frac{1}{m} \int_{Cos} \vec{r}\, dm \tag{3.6}$$

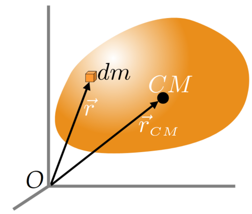


Fig. 3.8: CM of a continuous body

If the body is a **rigid body**, the CM is a point attached to the body. If the body performs only **translational motions**, the motion of the body is determined completely by the motion of the CM.

In practice, we will take into account that **if the motion of the body is only translational**, all points have the same velocity: **any point in the body is valid to for describing the velocity of** CM.

CM of volumes, surfaces and curves

Generally, a body (an asteroid, for example) will extend in three dimensions of space, in addition to occupying a volume. Despite having three dimensions, the body can in many cases be treated as a surface, such as a parabolic communication antenna. In other cases, although the body extends over all three dimensions, it can be treated as a curve, such as a helical spring. In Table 3.1 we can see how these cases can be treated in terms of mass density. ρ, σ and μ are volume, surface and linear mass densities, respectively.

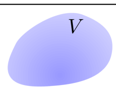
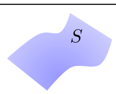
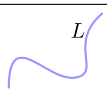
	$dm = \rho\, dv \Rightarrow$	$\vec{r}_{CM} = \frac{1}{\int \rho\, dv} \int \vec{r} \rho\, dv$	$\overset{\text{Hom}}{=} \frac{1}{V} \int \vec{r}\, dv$
	$dm = \sigma\, ds \Rightarrow$	$\vec{r}_{CM} = \frac{1}{\int \sigma\, ds} \int \vec{r} \sigma\, ds$	$\overset{\text{Hom}}{=} \frac{1}{S} \int \vec{r}\, ds$
	$dm = \mu\, d\ell \Rightarrow$	$\vec{r}_{CM} = \frac{1}{\int \mu\, d\ell} \int \vec{r} \mu\, d\ell$	$\overset{\text{Hom}}{=} \frac{1}{L} \int \vec{r}\, d\ell$

Table 3.1: CM of continuous bodies: volumes, surfaces and lines

The density of a **homogeneous** body is constant. For the homogeneous bodies, the CM coincides with the **geometric centre or centre of symmetry** CS(see Figure 3.9). If the body has enough elements of symmetry, it is easy to find the CS. In this case, the CS is the point that is common to all elements of symmetry (axes, planes, point).

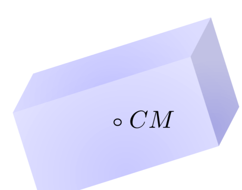


Fig. 3.9: The CM of a homogeneous body coincides with the geometric center or centre of symmetry CS

If a homogeneous bulky body of density ρ has the shape of a thick surface of constant thickness h and its curvature is negligible, we can treat it as a body surface of σ density. To find σ in terms of ρ, we only need to take into account that a piece of surface S of the body will have a volume $V = Sh$ and a mass that we can find by means of ρ or σ. Thus, $m = \rho V = \sigma S$; as a result, $\sigma = \rho h$.

If a homogeneous bulky body, of density ρ, has the shape of a thick curve of constant section s and its curvature is negligible, we can treat it as a linear body of density μ. To find μ in terms of ρ, we only need to take into account that a piece of the body of length L will have a volume $V = Ls$ and a mass that we can find by means of ρ or μ. Thus, $m = \rho V = \mu L$; as a result, $\mu = \rho s$.

CM of simple bodies

Here, simple means we know its CS or, in general, its CM. The table on page 337 shows the expressions for finding the centre of symmetry of some homogeneous bodies that we can use throughout the book. It does not include cases in which the CS is determined totally by symmetry, such as the circle, the sphere, the rectangle, etc.

CM of composite bodies

Consider a set of simple bodies, C_i, $i = 1...N$, of known masses m_i and positions of the CM, $\vec{r}_{CM}^{(i)}$ forming a composite body (see Figure 3.10). The position of the CM is:

$$\vec{r}_{CM} = \frac{1}{m}\sum_{i=1}^{N} m_i\,\vec{r}_{CM}^{(i)} \tag{3.7}$$

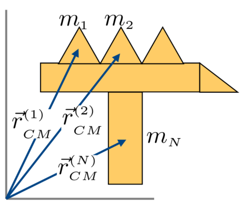


Fig. 3.10: The CM of a composite body can be expressed, by means of a convenient decomposition into simple bodies, in terms of masses m_i and positions $\vec{r}_{CM}^{(i)}$ of the CM of each simple body

Proof. As we know m_i and $\vec{r}_{CM}^{(i)}$ and we know that $\vec{r}_{CM}^{(i)} = \frac{1}{m_i}\int_{C_i}\vec{r}\,dm$, we can obtain each of the integrals $\int_{C_i}\vec{r}\,dm$

$$\int_{C_i}\vec{r}\,dm = m_i\,\vec{r}_{CM}^{(i)} \tag{3.8}$$

The expression for the CM position (3.6) of the composite body is considered as a single continuous body and it is extended to the domain of integration that encompasses the whole body C, i.e. $C = C_1 \cup ... \cup C_N$ where C_i is the domain of each body:

$$\vec{r}_{CM} = \frac{1}{m}\int_{C_1\cup...\cup C_N}\vec{r}\,dm$$

If we treat the integral extended to the integration domain $C_1 \cup ... \cup C_N$ as a sum of integrals in each integration subdomain C_i and substitute the integrals according to (3.8), we have

$$\vec{r}_{CM} = \frac{1}{m}\left(\int_{C_1}\vec{r}\,dm + ... + \int_{C_N}\vec{r}\,dm\right) = \frac{1}{m}\left(m_1\,\vec{r}_{CM}^{(1)} + ... + m_N\,\vec{r}_{CM}^{(N)}\right)$$

i.e.,

$$\vec{r}_{CM} = \frac{1}{m}\sum_{i=1}^{N} m_i\,\vec{r}_{CM}^{(i)}$$

■

If we interpret the masses m_i as $m_i > 0 \Rightarrow$ to add, and $m_i < 0 \Rightarrow$ to subtract, we can treat a body composed of bodies and of simple holes (see Figure 3.11).

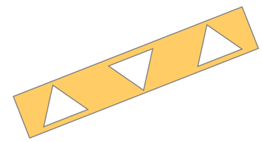


Fig. 3.11: We can interpret the sum of masses in such a way that the positive sign means to "add mass" and the negative sign means to "remove mass"

Problem 3.2.1. The following table shows the positions and velocities, at a given instant, of a three-particle system.

i	1	2	3
m_i(kg)	2	3	5
$\vec{r}_i$(m)	$(-10,-10)$	$(30,10)$	$(10,20)$
$\vec{v}_i$(m/s)	$(10,30)$	$(-20,-10)$	$(10,-10)$

Table for Problem 3.2.1: masses, positions and velocities

In this instant, determine:

a) The position of the system's centre of mass.

b) The velocity of the system's centre of mass.

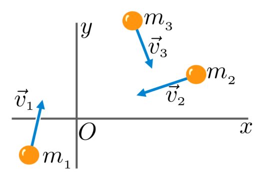
Solution

a)

$$\vec{r}_{CM} = \frac{\sum_{i=1}^{3} m_i \vec{r}_i}{m} = \frac{2(-10,-10)+3(30,10)+5(10,20)}{10} = (12,11)\,\text{m}$$

b)

$$\vec{v}_{CM} = \dot{\vec{r}}_{CM} = \frac{\sum_{i=1}^{3} m_i \dot{\vec{r}}_i}{m} = \frac{2(10,30)+3(-20,-10)+5(10,-10)}{10} = (1,-2)\,\text{m/s}$$



Solution to Problem 3.2.1

■

Problem 3.2.2. Given the flat homogeneous sheet shown in the figure, obtain the centre of mass.

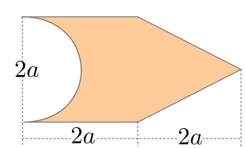
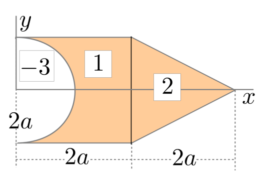


Figure for Problem 3.2.2

Solution

It is a homogeneous plane surface. Because it is symmetrical, we have $y_{CM} = 0$. Also, since $\sigma = \text{ct}$, instead of masses m_i we can use surfaces S_i. Thus, we decompose the sheet into three pieces: piece 1 is a square sheet; piece 2 is a triangle sheet; and piece 3 is a hole sheet. See the figure. We have

$$x_{CM} = \frac{1}{m}\sum_{i=1}^{N} m_i\, x_{CM}^{(i)} = \frac{1}{\sigma S}\sum_{i=1}^{N} \sigma S_i\, x_{CM}^{(i)} = \frac{1}{S}\sum_{i=1}^{N} S_i\, x_{CM}^{(i)}$$



Solution to Problem 3.2.2

Remember that the negative sign means it is a hole!

$$\left.\begin{array}{l} x_{CM} = \frac{1}{S_1+S_2-S_3}\left\{S_1\, x_{CM}^{(1)} + S_2\, x_{CM}^{(2)} - S_3\, x_{CM}^{(3)}\right\} \\ S_1 = 4a^2;\quad S_2 = \frac{1}{2}2a\; 2a\;;\quad S_3 = \frac{1}{2}\pi a^2 \\ x_{CM}^{(1)} = a;\quad x_{CM}^{(2)} = 2a + \frac{1}{3}2a;\quad x_{CM}^{(3)} = \frac{4a}{3\pi} \end{array}\right\} \Rightarrow \quad x_{CM} = 1.96a$$

■

3.3 Momentum

The **momentum $\vec{P}$ of a particle system** is defined as the sum of all particle momentums.

$$\vec{P} = \sum_{i=1}^{N} \vec{p}_i = \sum_{i=1}^{N} m_i \vec{v}_i = m\vec{v}_{CM} \tag{3.9}$$

The CM equation can be written as

$$\vec{F} = \frac{d\vec{P}}{dt} \tag{3.10}$$

➔ ***Conservation of momentum theorem***. *If the net force of a system of particles is zero, the momentum $\vec{P}$ remains constant over time.*

$$\frac{d\vec{P}}{dt} = 0 \tag{3.11}$$

This can also be written using the integrated expression between the two instants t_{ini} and t_{fi}

$$\vec{P}_{\text{ini}} = \vec{P}_{\text{fi}} \tag{3.12}$$

In usual parlance, it is said that *the momentum is conserved*. Note that as long as no external forces act on the system, the momentum is conserved.

Problem 3.3.1. A cobra of length L and uniformly distributed mass m lies stretched out on the ground. The cobra decides to rise vertically with a uniform velocity v (the tip of the tail does not move at any time). What is the reaction force of the ground on the part of the body in contact with it? Give the result as a function of L, m, v and the gravitational field g.

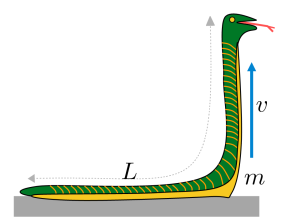
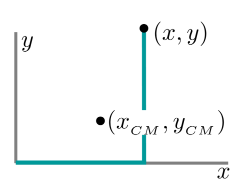


Figure for Problem 3.3.1

Solution

Let x be the portion of the snake in contact with the ground and y the vertical part. Note that both x and y will vary as the snake rises. These are functions of time t (see the figure). At any instant t, the total length of the snake is L; thus, $x + y = L$ and, since this expression is valid for all t, $\dot{x} + \dot{y} = 0$. If, in addition, we take into account that v is the constant velocity in the direction of y, we have the relations

$$\dot{y} = v \; ; \quad \dot{x} = -v \quad ; \; \ddot{x} = \quad \ddot{y} = 0 \tag{3.13}$$



Solution to Problem 3.3.1

We can calculate the position of the centre of mass, for all t, as a function of x and y. At any instant, the snake is a linear body formed by two straight homogeneous segments of length x and y, therefore,

$$\vec{r}_{CM} = (x_{CM}, y_{CM}) = \frac{1}{L}\left(\frac{x}{2}x + xy, \frac{y}{2}y\right) \tag{3.14}$$

If we derive (3.14) with respect to time and take into account (3.13),

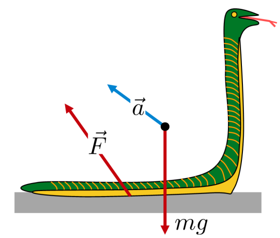
$$\dot{\vec{r}}_{CM} = \frac{1}{L} y\, v\, (-1, 1) \tag{3.15}$$

The momentum of the system can be written as

$$\vec{P} = m\dot{\vec{r}}_{CM} = \frac{m}{L} y\, v\, (-1, 1) \tag{3.16}$$

The net forces acting on the snake will be due to the weight and the reaction force of the ground, $\vec{F}$ i.e., $m(0, -g) + \vec{F}$ (see Figure). Thus, we have

$$\frac{d\vec{P}}{dt} = \frac{m}{L} v^2 (-1, 1) = m(0, -g) + \vec{F} \Rightarrow \vec{F} = \left(-\frac{m}{L} v^2, mg + \frac{m}{L} v^2\right) \quad \blacksquare$$



Solution to Problem 3.3.1

3.4 Angular momentum

The **angular momentum $\vec{L}_{(A)}$ of a system of particles with respect to a fixed point A** (see Figure 3.12) is defined as the sum of the angular momentum of each particle with respect to this point

$$\vec{L}_{(A)} = \sum_{i=1}^{N} \vec{L}_{(A)i} = \sum_{i=1}^{N} \vec{r}_{i(A)} \times m_i \vec{v}_i \tag{3.17}$$

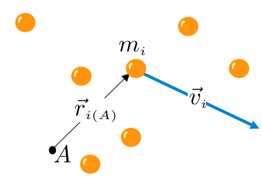


Fig. 3.12: Angular momentum of a system of particles with respect to a fixed A point

Then we take the time derivative:

$$\frac{d\vec{L}_{(A)}}{dt} = \sum_{i=1}^{N} \left(\dot{\vec{r}}_{i(A)} \times m_i \vec{v}_i + \vec{r}_{i(A)} \times m_i \vec{a}_i\right) \tag{3.18}$$

Now, taking into account that point A is fixed, $\dot{\vec{r}}_{(A)i} = \vec{v}_i$, the first vector product of (3.18) becomes zero. For the second vector product, we will take into account the equations of motion for the N particles, $m_i \vec{a}_i = \vec{F}_i + \sum_{j=1}^{N} \vec{F}_{ji}$. Thus, we have

$$\frac{d\vec{L}_{(A)}}{dt} = \sum_{i=1}^{N} \left(\vec{r}_{i(A)} \times \left[\vec{F}_i + \sum_{j=1}^{N} \vec{F}_{ji}\right]\right) \tag{3.19}$$

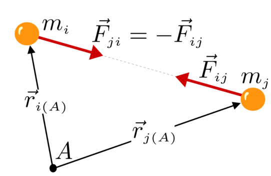


Fig. 3.13: Net torque of the internal forces

Assuming that the action–reaction forces have the same line of action, the net torque of the internal forces cancels out (see Figure 3.13):

$$\vec{r}_{i(A)} \times \vec{F}_{ji} + \vec{r}_{j(A)} \times \vec{F}_{ij} = (\vec{r}_{j(A)} - \vec{r}_{i(A)}) \times \vec{F}_{ij} = 0$$

Therefore, we obtain

$$\frac{d\vec{L}_{(A)}}{dt} = \vec{M}_{(A)} \tag{3.20}$$

where $\vec{M}_{(A)}$ is the net torque of the forces. We observe that only external forces are involved in $\vec{M}_{(A)}$

$$\vec{M}_{(A)} = \sum_{i=1}^{N} \vec{r}_{i(A)} \times \vec{F}_i \tag{3.21}$$

Conservation of angular momentum theorem. *If the net torque of the external forces with respect to point A is zero, $\vec{M}_{(A)} = 0$, the angular momentum $\vec{L}_{(A)}$ remains constant over time:*

$$\frac{d\vec{L}_{(A)}}{dt} = 0 \tag{3.22}$$

Using the integrated expression between two instants t_{ini} and t_{fi} it can be written as

$$\vec{L}_{(A)\text{ini}} = \vec{L}_{(A)\text{fi}} \tag{3.23}$$

In usual parlance, it is said that *the angular momentum is conserved*.

If the particle system moves **only by translation**, $\vec{v}_i = \vec{v}$, the angular momentum of the system, $\vec{L}_{(A)}$, can be expressed as

$$\vec{L}_{(A)} = \sum_{i=1}^{N} \vec{r}_{i(A)} \times m_i \vec{v}_i = \left(\sum_{i=1}^{N} \vec{r}_{i(A)} m_i \right) \times \vec{v} = \vec{r}_{CM(A)} \times \vec{P} \tag{3.24}$$

Generally, if we use $\vec{r}_{i(A)} = \vec{r}_{CM(A)} + \vec{r}_{i(CM)}$, we can write the angular momentum with respect to any fixed point A as

$$\vec{L}_{(A)} = \vec{L}_{(CM)} + \vec{r}_{CM(A)} \times \vec{P} \tag{3.25}$$

The expressions (3.20) and (3.21) are also valid if $A = CM$, although CM is in motion. Care must be taken to not take moving points $A \neq CM$. In this section it is better to consider A fixed. Later, when we study the rigid body, we will prove and use $A = CM$.

Problem 3.4.1. In the following table, we have the positions and the velocities of a three-particle system at a given instant.

i	1	2	3
m_i(kg)	2	3	5
$\vec{r}_i$(m)	$(-10,-10)$	$(30,10)$	$(10,20)$
$\vec{v}_i$(m/s)	$(10,30)$	$(-20,-10)$	$(10,-10)$

Table for Problem 3.4.1: masses, positions and velocities

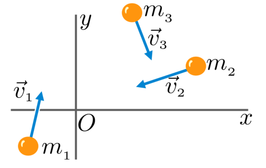
At this instant, determine:

a) The total momentum.

b) The total angular momentum with respect to the origin.

Solution

a) From Problem 3.2.1, we have $\vec{v}_{CM} = (1,-2)$ m/s and, thus, $\vec{P} = m\vec{v}_{CM} = 10(1,-2) = (10,-20)\,\text{kg m/s}$



Solution to Problem 3.4.1

b) Be careful! We cannot use $\vec{L} = \vec{r}_{CM} \times \vec{P}$, because the velocities of the particles are different. We have to use $\vec{L} = \sum_{i=1}^{3} \vec{L}_i$:

$$\vec{L}_1 = \vec{r}_1 \times \vec{p}_1 = 2 \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -10 & -10 & 0 \\ 10 & 30 & 0 \end{vmatrix} = -400\hat{k}$$

$$\vec{L}_2 = \vec{r}_2 \times \vec{p}_2 = 3 \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 30 & 10 & 0 \\ -20 & -10 & 0 \end{vmatrix} = -300\hat{k}$$

$$\vec{L}_3 = \vec{r}_3 \times \vec{p}_3 = 5 \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 10 & 20 & 0 \\ 10 & -10 & 0 \end{vmatrix} = -1500\hat{k}$$

We obtain $\vec{L} = \sum_{i=1}^{3} \vec{L}_i = \boxed{-2200\,\hat{k}\ \frac{\text{kg m}^2}{\text{s}}}$ ■

Problem 3.4.2. A truck transports a homogeneous rectangular box that is 2 m high and 1 m long. The friction coefficient of the box and truck is 0.4. Calculate:

a) The maximum acceleration that the truck can have without the box slipping (if we know that before the box does not overturn).

b) The maximum acceleration that the truck can have without the box overturning if we attach the box to the truck by a frictionless axle A.

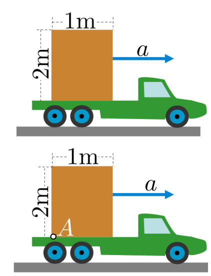
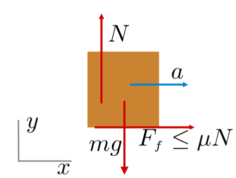


Figure for Problem 3.4.2

Solution

a) $\vec{F} = m\vec{a} \Rightarrow (\mu N, N - mg) = m(a, 0) \Rightarrow a = \mu g = 0.4 \times 9.8 = \boxed{3.92\,\text{m/s}^2}$



Solution to Problem 3.4.2

b) We use $\frac{d\vec{L}_{(O)}}{dt} = \vec{M}_{(O)}$ with the angular momentum and torque calculated with respect to the origin (see the figure). We can use the corkscrew rule.

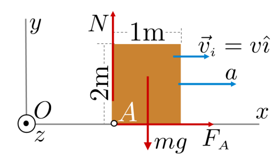
For the angular momentum, note that the motion is translational. It is enough to use the velocity vector at the CM (see the figure). We obtain $\vec{L}_{(O)} = -1mv\,\hat{k}$. For the moment of force, we analyse the instant in which the box is about to overturn. At this instant, the normal passes through point A, since the only contact between the box and the truck is at point A. We obtain

$$\vec{M}_{(O)} = -\left\{(\overline{OA} + 0.5)mg - \overline{OA}\,N\right\}\hat{k} = -0.5\,mg\,\hat{k}$$

Thus, deriving with respect to time $\vec{L}_{(O)}$ and making it equal to the torque

$$-1ma\hat{k} = -0.5\,mg\,\hat{k}$$

Therefore, $a = 0.5g = \boxed{4.9\,\text{m/s}^2}$ ■



Solution to Problem 3.4.2

3.5 Work, kinetic energy and potential energy

Work

The concept of work, according to Interpretation **1** of Section 2.4, can be applied here for each force separately. The same must be done with the concept of conservative force and potential energy associated with a force.

In order to define the concept of energy for a system, we extend Interpretation **2** of Section 2.4 to all particles and forces in the system.

We call the ***configuration*** of the system at a given instant the set of positions for all particles.

The work W performed by all the forces on each particle, $\vec{F}_i + \sum_{j=1}^{N} \vec{F}_{ji}$, when particles go from one configuration $P_1 = \{P_{i1}\}$ to another $P_2 = \{P_{i2}\}$ through a path $C = \{C_i\}$, is (see Figure 3.14)

$$W = \sum_{i=1}^{N} \int_{C:P_1}^{P_2} \left(\vec{F}_i + \sum_{j=1}^{N} \vec{F}_{ji} \right) d\vec{r}_i \qquad (3.26)$$

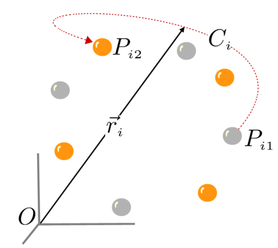


Fig. 3.14: The system goes from configuration P_1 to configuration P_2. There are many paths $C = \{C_i\}$ that go from one configuration to the other

Kinetic energy or ability to work due to speed

Using a similar calculation to the one in Section 2.4 leads us to the kinetic energy concept

$$\begin{aligned} W &= \sum_{i=1}^{N} \int_{C:P_1}^{P_2} \left(\vec{F}_i + \sum_{j=1}^{N} \vec{F}_{ji} \right) \cdot d\vec{r}_i = \sum_{i=1}^{N} \int_{C:P_1}^{P_2} m_i \vec{a}_i \cdot d\vec{r}_i \\ &= \ldots = \int_{P_1}^{P_2} d \left(\sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 \right) \end{aligned}$$

Consequently, we define the **kinetic energy** E_c **of a system of** N **particles** as

$$E_c = \sum_{i=1}^{N} \frac{1}{2} m_i \, v_i^2 \qquad (3.27)$$

and we can state the following theorem:

➔ ***Work-energy theorem***:

$$W = \Delta E_c \qquad (3.28)$$

This is also known as the **theorem live forces** and as the **kinetic energy theorem**.

Potential energy or ability to work due to position

The forces on each particle are the external forces and the interaction forces between particles: $\vec{F}_i + \sum_{j=1} \vec{F}_{ji}$. If all the forces are conservative, they will have potential energies U_i, U_{ij} that fulfil the relationships

$$\vec{F}_i = -\frac{\partial U_i}{\partial \vec{r}_i} \; ; \; \vec{F}_{ji} = -\frac{\partial U_{ji}}{\partial \vec{r}_i}, \quad U_{ii} = 0 \tag{3.29}$$

In addition to the forces being conservative, and as a consequence of the action-reaction principle, the forces depend only on the position of the particles in accordance with $\vec{F}_{ji} = f_{(ji)}(\vec{r}_i - \vec{r}_j)$, where $f_{(ji)} = f_{(ij)}$ depends only on the distance between particles i and j[1]. This allows choosing U_{ij} which satisfies (3.29), depends only on the distance between particles and is symmetric $U_{ji} = U_{ij}$. We denote this as **(potential) energy of interaction**.

[1] Looking at the gravitational and electrostatic cases we see that they meet these conditions

If the system goes from one position configuration P_1 to another one, P_2, through a path C (see Figure 3.14), the work it does is

$$W = \sum_{i=1}^{N} \int_{C:P_1}^{P_2} \left(\vec{F}_i + \sum_{j=1} \vec{F}_{ji}\right) \cdot d\vec{r}_i = \sum_{i=1}^{N} \int_{C:P_1}^{P_2} \vec{F}_i \cdot d\vec{r}_i \; + \; \sum_{i=1}^{N}\sum_{j=1}^{N} \int_{C:P_1}^{P_2} \vec{F}_{ji} \cdot d\vec{r}_i \tag{3.30}$$

➔ *The work of the conservative forces is equal to the decrease in potential energy:*

$$W = -\Delta U \tag{3.31}$$

with the potential energy U for the system of N particles being defined as

$$U = \sum_{i=1}^{N} U_i + \sum_{i=1}^{N}\sum_{i<j}^{N} U_{ij} \tag{3.32}$$

Proof. The first term of the right side of (3.30) can be written as

$$\sum_{i=1}^{N} \int_{C:P_1}^{P_2} \vec{F}_i \cdot d\vec{r}_i = -\sum_{i=1}^{N} \int_{:P_1}^{P_2} dU_i \tag{3.33}$$

Concerning the second term of the right side, we have to be much more careful. First, we exchange i for j, $\sum_{i=1}^{N}\sum_{j=1}^{N} \int_{C:P_1}^{P_2} \vec{F}_{ij} \cdot d\vec{r}_j$, and add the result to the original term. Since the added term is the same as the old one, we divide the sum by 2

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \int_{C:P_1}^{P_2} \vec{F}_{ji} \cdot d\vec{r}_i = \frac{1}{2} \sum_{i=1}^{N}\sum_{j=1}^{N} \int_{C:P_1}^{P_2} (\vec{F}_{ij} \cdot d\vec{r}_j + \vec{F}_{ji} \cdot d\vec{r}_i) = \dots \tag{3.34}$$

Now we substitute the forces as a function of their potentials according to (3.29)

$$... = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{C:P_1}^{P_2}(\frac{\partial U_{ij}}{\partial \vec{r}_j}\cdot d\vec{r}_j + \frac{\partial U_{ji}}{\partial \vec{r}_i}\cdot d\vec{r}_i) = ... \tag{3.35}$$

Remember that $U_{ji} = U_{ij}$. In addition, we will take into account that U_{ij} depends only on $\vec{r}_i$ and $\vec{r}_j$ and, therefore, the differential is $dU_{ij} = \frac{\partial U_{ij}}{\partial \vec{r}_j}\cdot d\vec{r}_j + \frac{\partial U_{ij}}{\partial \vec{r}_i}\cdot d\vec{r}_i$. We can write (3.35) as

$$... = -\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int_{C:P_1}^{P_2} dU_{ij} = ... \tag{3.36}$$

Finally, in order to not add any more terms than necessary, we restrict the sum of j to only terms $i < j$, and thus the factor $\frac{1}{2}$ disappears.

$$... = -\int_{C:P_1}^{P_2} d\left(\sum_{i=1}^{N}\sum_{i<j}^{N} U_{ij}\right) \tag{3.37}$$

Considering the above expressions (3.30, 3.33-3.37), the potential energy U of the system of N particles can be defined as (3.32), and we obtain (3.31). ■

Mechanical energy

The **mechanical energy E of a conservative system of N particles** is defined as

$$E = E_c + U \tag{3.38}$$

Deriving with respect to time and following steps that are similar to those seen in Section 2.4, we obtain the following, as expected.

➔ ***Conservation of mechanical energy theorem***. *The numerical value of mechanical energy in a conservative system of N particles remains constant over time.*

$$\frac{dE}{dt} = 0 \tag{3.39}$$

It can also be written using the integrated expression between two instants t_{ini} and t_{fi}

$$E_{\text{ini}} = E_{\text{fi}} \tag{3.40}$$

In usual parlance, it is said that the *mechanical energy is conserved*.

If there are non-conservative forces (NC), these are not included in the energy (they have no associated potential energy). However, the following can be applied.

➔ ***Mechanical energy theorem***. *The work of the non-conservative forces is equal to the increase in the mechanical energy of the system:*

$$W_{NC} = \Delta E \tag{3.41}$$

We observe that energy can still be conserved if we restrict ourselves to motions for which the NC forces "do not work", that is, $W_{NC} = 0$. Later, when we deal with constrained systems, we will use this condition.

As we will see later in Section 3.11 for the case of a **rigid body**, the work of the internal forces is zero; so, $W = \Delta E_c$, where W is the work of the external forces. If, in addition, the motion is only **translation**, $v_i = v_{CM}$ and we have $E_c = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2 = \frac{1}{2} m v_{CM}^2$. Thus, the mechanical energy of a **rigid body restricted to translational motions** is

$$E = \frac{1}{2} m v_{CM}^2 + U \quad (3.42)$$

where, in this case, U is the sum of potential energies of the body's conservative external forces.

Problem 3.5.1. Two particles of masses 2 kg and 3 kg are bound together with a rope running through a spring, as seen in the figure. Both the spring and the rope have negligible mass. The spring has a constant of 12000 N/m and is compressed to a length of 10 cm. We cut the rope. Find the velocity of each particle.

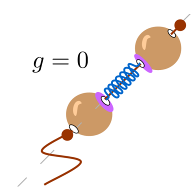
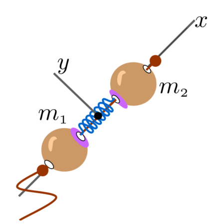


Figure for Problem 3.5.1

Solution

For convenience, we define $m_1 = 2\,\text{kg}$, $m_2 = 3\,\text{kg}$, $k = 12000\,\text{N/m}$ and $L = 0.1\,\text{m}$.

The causes of the motion are aligned along the x-axis; so, we can apply the symmetry principle (see the Section 1.4) to deduce that the velocities of both particles will be aligned with the x-axis: $\vec{v}_1 = (v_1, 0, 0)$ and $\vec{v}_2 = (v_2, 0, 0)$.



Solution to Problem 3.5.1

Applying the momentum conservation between the instants just before and just after cutting the rope, we have:

$$0 = m_1 \vec{v}_1 + m_2 \vec{v}_2 \Rightarrow 2v_1 + 3v_2 = 0$$

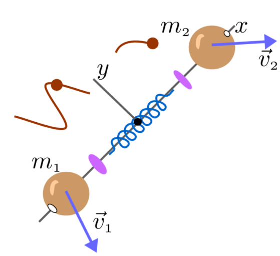
We can do the same with energy:

$$\frac{1}{2} k L^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \Rightarrow 60 = v_1^2 + \frac{3}{2} v_2^2$$

from which we get $v_1 = \pm 6$ and $v_2 = \mp 4$. Using the fact that the spring decompresses (this fact has not been used yet!), the signs are determined and we obtain

$$\vec{v}_1 = (-6, 0, 0)\,\text{m/s} \;;\; \vec{v}_2 = (4, 0, 0)\,\text{m/s}$$

■



Solution to Problem 3.5.1

Problem 3.5.2. The following table shows the positions and velocities of a three-particles system at a given instant.

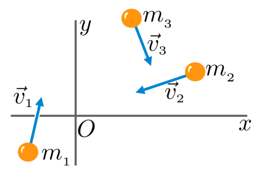
At this instant, determine:

i	1	2	3
m_i(kg)	2	3	5
$\vec{r}_i$(m)	$(-10,-10)$	$(30,10)$	$(10,20)$
$\vec{v}_i$(m/s)	$(10,30)$	$(-20,-10)$	$(10,-10)$

Table for Problem 3.5.2: masses, positions and velocities

a) The kinetic energy of the system.

b) The kinetic energy associated with the motion of the centre of mass (all the mass concentrated in the CM together with its velocity)



Solution to Problem 3.5.2

Solution

a)

$$E_c = \frac{1}{2}\sum_{i=1}^{3} m_i v_i^2 = \frac{1}{2}\left\{2(10^2+30^2)+3(20^2+10^2)+5(10^2+10^2)\right\} = \boxed{2250\,\mathrm{J}}$$

b) From Problem 3.2.1, we have $\vec{v}_{CM} = (1,-2)$ m/s. The kinetic energy of a particle of mass $m = 10$ kg moving with the velocity of the CM is

$$\frac{1}{2} m\, v_{CM}^2 = \frac{1}{2} 10 \cdot (1^2+2^2) = \boxed{25\,\mathrm{J}}$$

which generally does not match the kinetic energy of the system. ■

3.6 Collisions

A **collision** is a process in which two or more particles with very short range interaction, meet at a point and at an instant and, consequently, modify their velocities (see Figure 3.15).

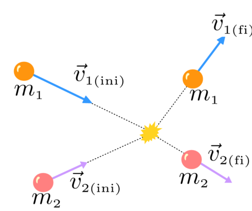


Fig. 3.15: Forces exist only at the instant of collision

We will restrict ourselves to two particles while considering two instants in which the particles are far enough apart and do not **interact**: the initial instant (ini), just before the collision; and the final instant (fi), just after the collision. We will assume that the external forces are much weaker than those caused by the collision, such that the external impulse between (ini) and (fi) is negligible.

We will also restrict ourselves to **frictionless collisions**. This means that when contact occurs between the two particles, which we can visualize as spherical bodies (see Figures 3.16 or 3.17), there are no forces in the tangent to the contact plane or **collision plane** and, therefore, there is no exchange of angular momentum. If the particles or bodies were to had an initial rotation, they would maintain this rotation after the collision.

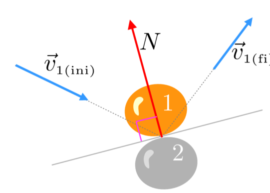


Fig. 3.16: Collision plane

We can apply the theorem of conservation of total momentum, giving us:

$$\vec{P}_{(\mathrm{ini})} = \vec{P}_{(\mathrm{fi})} \tag{3.43}$$

Although we do not know the geometry of the collision (the collision plane, see the Figure 3.16), we will be able to treat the collisions totally elastic and totally inelastic.

➔ **Totally elastic collisions**. These are collisions in which the energy is conserved, $E_{(\text{ini})} = E_{(\text{fi})}$. We can write this in terms of kinetic energy, since before or after the collision either no forces exist or they are negligible. Thus, for totally elastic collisions,

$$E_{c(\text{ini})} = E_{c(\text{fi})} \tag{3.44}$$

The condition for an elastic collision and the consequent conservation of energy require an explanation. One may think than an *elastic* ball will conserve its energy when it bounces on the ground, which is essentially true as long as we account for all the terms of the ball's kinetic energy. Before the collision, the ball has translational energy. Once the collision has taken place, the ball will have translational kinetic energy as well as internal vibration energy, due to the elasticity of the material. It is only when the internal vibration energy is negligible that we can say the collision is elastic in the sense that we mean here in this chapter, namely that the colliding objects do not accumulate vibration energy. Thus, the elastic collision is energy conservation without considering the terms of internal energy, as in the case of vibration.

➔ **Totally inelastic collisions**:

Implosion. Both particles join togehter (implode) and a single particle is formed after the collision, $\vec{v}_{1(\text{fi})} = \vec{v}_{2(\text{fi})}$.

Explosion. A single particle explodes and becomes two particles, $\vec{v}_{1(\text{ini})} = \vec{v}_{2(\text{ini})}$.

➔ **Partially elastic collisions**. Taking into account the particles contact geometry, the collision plane (see Figure 3.16 or 3.17), and the fact that the collision is frictionless (that is, the only acting force at the moment of the collision is normal to the collision plane), then only the momentum component in the normal direction will change while the components tangent to the collision plane will be conserved. Thus, if the components tangential to the collision plane are indicated by $||$:

$$\vec{p}_{i||(\text{ini})} = \vec{p}_{i||(\text{fi})} \Rightarrow \vec{v}_{i||(\text{ini})} = \vec{v}_{i||(\text{fi})} \quad i = 1, 2 \tag{3.45}$$

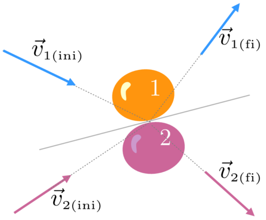


Fig. 3.17: e is a relationship between normal relative velocities and the collision plane

If the collision is completely elastic or completely inelastic, we can use the above conditions for energy conservation, explosion, or implosion. If the collision is partially elastic and the collision plane is known, the **coefficient of restitution** e is defined (see Figure 3.17). If we indicate with $\perp$ the normal component to the collision plane, the coefficient of restitution is

$$e = \left| \frac{v_{1\perp(\text{fi})} - v_{2\perp(\text{fi})}}{v_{1\perp(\text{ini})} - v_{2\perp(\text{ini})}} \right| \tag{3.46}$$

The coefficient of restitution e is a quantity that depends on the material of the bodies. It can be measured using (3.46).

➔ **Collisions with the ground**. We can also consider the collision of a particle with the ground or a wall (see Figure 3.18). In this case, there are external forces in the form of the ground reaction. We have $\vec{p}_{(\text{ini})} \neq \vec{p}_{(\text{fi})}$. If the collision is frictionless, the external force has only the normal component to the ground. As in the case of two particles, $\vec{p}_{||(\text{ini})} = \vec{p}_{||(\text{fi})} \Rightarrow \vec{v}_{||(\text{ini})} = \vec{v}_{||(\text{fi})}$

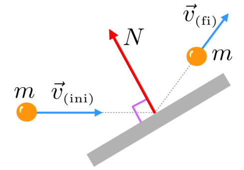


Fig. 3.18: Collision with a wall

If the collision is completely elastic, we will have $v^2_{\perp(\text{ini})} = v^2_{\perp(\text{fi})}$. If partially elastic, the coefficient of restitution can be defined according to

$$e = \left| \frac{v_{\perp(\text{fi})}}{v_{\perp(\text{ini})}} \right| \tag{3.47}$$

In this case, e depends on the constituent material of the body, the ground or the wall. It can be measured using (3.47).

Problem 3.6.1. Calculate the velocities after the collision of the balls in the figure (corresponding to the instant of the collision), which have equal masses. Bear in mind that they are smooth and their coefficient of restitution is 0.9.

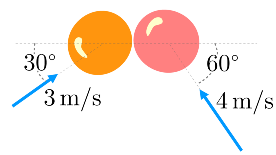


Figure for Problem 3.6.1

Solution

We use the notation B for before the collision and A for after the collision.

According to the data reflected in the figure, we have $\vec{v}_{B1} = (3\frac{\sqrt{3}}{2}, \frac{3}{2})$ and $\vec{v}_{B2} = (-2, 2\sqrt{3})$

From the conservation of the momentum parallel to the collision plane for each particle, we have

$$v_{yA1} = 1.5\,\text{m/s} \quad ; \quad v_{yA2} = 3.46\,\text{m/s} \tag{1}$$

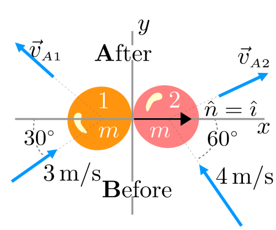
From the conservation of the total momentum, we have $m\vec{v}_{B1} + m\,\vec{v}_{B2} = m\vec{v}_{A1} + m\,\vec{v}_{A2}$.

The component of this equation in the y direction is automatically satisfied with the values found in (1). We take only the x component:

$$3\frac{\sqrt{3}}{2} - 2 = v_{xA1} + v_{xA2} \tag{2}$$

Knowing of the coefficient of restitution, we obtain

$$\pm 0.9 = \frac{v_{xA1} - v_{xA2}}{3\frac{\sqrt{3}}{2} - (-2)} \tag{3}$$



Solution to Problem 3.6.1

Now we have to solve equations (2) and (3), for which we choose the solution in which the two balls do not intersect, that is, $v_{xA2} - v_{xA1} > 0$

We obtain: $v_{xA1} = -1.77\,\text{m/s}$; $v_{xA2} = 2.36\,\text{m/s}$ ■

3.7 Gravitational and electromagnetic interaction

Law of universal gravitation

In 1687, Isaac Newton published his law of universal gravitation, which unified Kepler's laws. It also provided a more fundamental explanation of his three laws, which had been obtained empirically by direct observation of the motion of the planets around the Sun. Newton's law of gravitation is not a rule for predicting the motion of planets, but is instead a law obeyed by all bodies due to the fact that they have mass, whether they are planets, stones, any other object or particles (see Figure 3.19).

➔ ***Law of universal gravitation***

$$\vec{F}_{ij} = -G\frac{m_i m_j}{r_{ij}^2}\hat{r}_{ij} \tag{3.48}$$

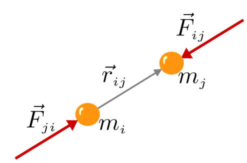


Fig. 3.19: Gravitational interaction between two masses

where $\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$, $G = 6.673\times 10^{-11}\ \frac{\mathrm{N\,m^2}}{\mathrm{kg^2}}$ is the universal gravitational constant, and m_i and m_j are the **gravitational masses** of each of the interacting particles.

Experiments have proven a highly remarkable fact, which is namely that, for a given particle, the gravitational mass (m in the equation for universal gravitation (3.48)) and the inertial mass (m in $\vec{F} = m\vec{a}$), which can be expressed in the same units, have the same numerical value. Newton saw that these were two different concepts in his theory and that their equality needed an explanation. Experimental verification of this equality is the germ that gave rise to Albert Einstein's **equivalence principle**, a relativity version of Newton's first law, which forms a part of the theory of general relativity. According to the equivalence principle, the inertial and gravitational mass of a particle are exactly equal because they are the same property.

If we take as an example the interaction between two protons, $m_p = 9.109\times 10^{-31}$ kg, separated by a typical atomic distance of $r = 1$ Å$= 10^{-10}$ m, we obtain from (3.48)

$$|F_G| = G\frac{m_p^2}{r^2} = 5.54\times 10^{-51}\ \mathrm{N} \tag{3.49}$$

The gravitational interaction energy between two particles is

$$U_{ij} = -G\frac{m_i m_j}{r_{ij}} \tag{3.50}$$

The gravitational interaction energy between three particles is

$$U = U_{12} + U_{13} + U_{23} = -G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}\right) \tag{3.51}$$

Coulomb's law

Coulomb's law was published in 1785 by Charles-Augustin de Coulomb, based on measurements he made of the attractive and repulsive forces between electric charges, for which he used a torsion balance (see Figure 3.21).

Fig. 3.20: Charles-Augustin de Coulomb (1737-1806) was a French physicist and military engineer

➔ *Coulomb's law*

$$\vec{F}_{ij} = K\frac{Q_i Q_j}{r_{ij}^2}\hat{r}_{ij} \tag{3.52}$$

where $K = 8.987 \times 10^9 \frac{\mathrm{N\,m^2}}{\mathrm{C^2}}$ is the constant of electrostatic interaction, and Q_i and Q_j are the electric charges of each of the interacting particles measured in the international system of units for an electric charge, the **coulomb** (C): $[Q] = \mathrm{C}$.

If we take as an example the interaction between two protons, $Q_p = 1.602 \times 10^{-19}$ C, separated by a typical atomic distance of $r = 1$Å$= 10^{-10}$ m, we obtain

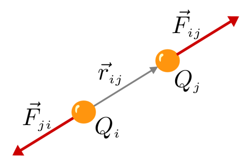


Fig. 3.21: Electrostatic interaction between two charges

$$|F_E| = K\frac{Q_p^2}{r^2} = 2.31 \times 10^{-8}\,\mathrm{N} \tag{3.53}$$

The electrostatic interaction energy between two particles is

$$U_{ij} = K\frac{Q_i Q_j}{r_{ij}} \tag{3.54}$$

the electrostatic interaction energy between three particles is

$$U = U_{12} + U_{13} + U_{23} = K\left(\frac{Q_1 Q_2}{r_{12}} + \frac{Q_1 Q_3}{r_{13}} + \frac{Q_2 Q_3}{r_{23}}\right) \tag{3.55}$$

Note that both of the described interaction laws are quasi-identical in terms of the geometry (the dependence of the forces on the positions). The only difference is that one is attractive and the other is repulsive. This means that, for identical particles, the gravitational interaction causes an attractive force and the electrostatic interaction causes a repulsive force. They differ in their charges and interaction constants. Comparing the results in the case of protons, the electromagnetic interaction is about 10^{43} times more intense than the gravitational interaction!

Problem 3.7.1. The following table lists the charges and positions of a three-particles system. Determine the total potential energy.

i	1	2	3
$Q_i(\mu\mathrm{C})$	-2	3	5
$\vec{r}_i(\mathrm{m})$	$(-10, -10)$	$(30, 10)$	$(10, 20)$

Table for Problem 3.7.1: charges and positions

Solution

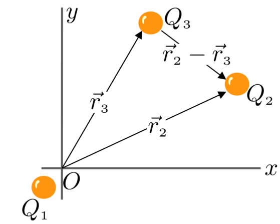
$$U = U_{12} + U_{13} + U_{23} = K\left(\frac{Q_1 Q_2}{r_{12}} + \frac{Q_1 Q_3}{r_{13}} + \frac{Q_2 Q_3}{r_{23}}\right)$$

$$r_{12} = \sqrt{(\vec{r}_1 - \vec{r}_2)^2} = \sqrt{40^2 + 20^2} = 44.72$$

$$r_{13} = \sqrt{(\vec{r}_1 - \vec{r}_3)^2} = \sqrt{20^2 + 30^2} = 36.05$$

$$r_{23} = \sqrt{(\vec{r}_2 - \vec{r}_3)^2} = \sqrt{20^2 + 10^2} = 22.36$$

$$U = 9\times\ 10^9 \left(\frac{-2\times 3}{44.72} + \frac{-2\times 5}{36.05} + \frac{3\times 5}{22.36}\right) 10^{-12} = 2.33\times 10^{-3}\ \text{J} \quad \blacksquare$$



Solution to Problem 3.7.1

3.8 Constraints and reactions. Possible displacements and virtual displacements. Ideal reactions

A system of N particles without constraints requires $3N$ data to express their position. Thus, we say that it has $3N$ **degrees of freedom**. If the particles are constrained, the number of degrees of freedom is smaller. Let us look at some examples.

Simple pendulum. A particle is forced to move on the surface of a sphere of radius ℓ (see Figure 3.22). For this pendulum, instead of a rope that can become slack, we have a rigid rod of negligible mass. At one end is a particle and at the other is a pivot point. The system has two degrees of freedom.

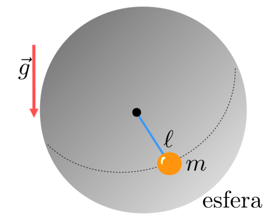


Fig. 3.22: The particle is forced to move by the surface of a sphere

Rigid body. This is made up of a set of constrained particles forced to keep their relative distances constant (see Figure 3.23). Thus, the rigid body will have six degrees of freedom, since the positions of all particles are completely determined with respect to a reference position by the body translation (three degrees of freedom) and rotation (three degrees of freedom) (see Figure 3.24).

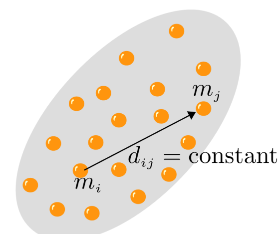


Fig. 3.23: The particles keep fixed the distances between them

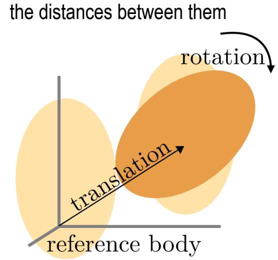


Fig. 3.24: The positions of the particles are determined if we know the translation and the rotation of the body with respect to the reference body

Forced pendulum. This pendulum is like the previous one but the sphere is now forced to move horizontally by means of an actuator (see Figure 3.25). In this case, the actuator is a hand that moves the pivot at will. The motion of the actuator is known: $x(t)$. It also has two degrees of freedom.

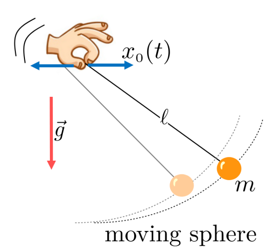


Fig. 3.25: We move the pivot at will

Constraints

Each particle i in the system is characterized by its mass and its vector position

$$\{m_i, \vec{r}_i\}, \quad \vec{r}_i = (x_i, y_i, z_i), \quad i = 1...N$$

Positions may be subject to constraints. We consider only **geometric constraints**, that is, they do not depend on velocities, although they may depend on time. In the latter case, the constraint is called an ***actuator***. If we have L degrees of freedom, it is because we have $3N - L$ independent constraints. Generally, geometric constraints can be expresses as:

$$f_a(\vec{r}_i, t) = 0 \quad , a = 1...3N - L \tag{3.56}$$

It is very convenient to express the constraints parametrically. Naming the L parameters $q_1...q_L$, the same expression (3.56) can be written as

$$\vec{r}_i = \vec{r}_i(q_1...q_L, t) \tag{3.57}$$

Generally, we will consider the constraints as data for the problem.

Example. A particle is forced to move along a certain curve $f(x, y, z) = 0$ in space because it passes through a wire with the exact form of this curve. We can determine the constraint $f(x, y, z) = 0$ only by observing the shape of the wire.

If, for example, the curve is a circumference of radius ℓ in the (x, y) plane, the corresponding (3.56) expression will be

$$x^2 + y^2 - \ell^2 = 0$$

and the parametric form (3.57) can be written with a single parameter $q_1 = \theta$

$$x = \ell \, \cos\theta$$
$$y = \ell \, \sin\theta$$

If the wire is a straight line in the (x, y) plane, we will have the corresponding (3.56) expression

$$ax + b - y = 0$$

where a and b will be known constants. We can write the parametric form (3.57) by making $q_1 = q$:

$$x = q$$
$$y = aq + b$$

Possible and virtual displacements

Possible system displacements $d\vec{r}_i$ are compatible with the constraints. For example, in the case of the particle bound to the wire (see Figure 3.26), the possible displacement must follow the trajectory of the wire. In saying that possible displacements are compatible with the constraints, this means that carrying out $d\vec{r}_i$ displacements result in the constraint condition $f_a(\vec{r}_i, t) = 0$ continuing to be fulfilled for the new position. That is to say,

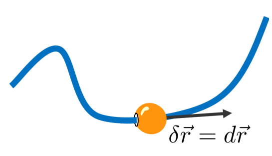


Fig. 3.26: We can determine the constraint only by observing the wire shape

$$df_a = \sum_{i=1}^{N} \frac{\partial f_a}{\partial \vec{r}_i} \cdot d\vec{r}_i + \frac{\partial f_a}{\partial t} dt = 0 \tag{3.58}$$

where we observe that we have also taken into account the dt variation, since the possible displacement takes some time to be carried out.

We can use the parametric form of the constraints to find the possible $d\vec{r}_i$ displacements in terms of the L q_a parameters:

$$d\vec{r}_i = \sum_{a=1}^{L} \frac{\partial \vec{r}_i}{\partial q_a} dq_a + \frac{\partial \vec{r}_i}{\partial t} dt \tag{3.59}$$

We define the $\delta\vec{r}_i$ **virtual displacements** of the system as displacements that are carried out without variation in time, i.e. by freezing time. If we imagine that we are filming a movie scene in which the pivot of an oscillating pendulum is moved by a hand (see Figure 3.27), then the possible displacement are what we are recording and can see later in the finished movie. Virtual displacement would correspond to stopping the recording, moving the particle without moving the hand and, at the end of this displacement, continue recording.

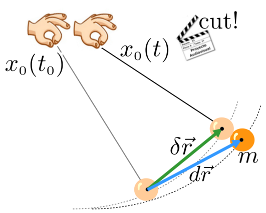


Fig. 3.27: The subtle difference between possible and virtual displacement

From the mathematical point of view, virtual displacements fulfil

$$\sum_{i=1}^{N} \frac{\partial f_a}{\partial \vec{r}_i} \cdot \delta\vec{r}_i = 0 \tag{3.60}$$

or, in parametric form,

$$\delta\vec{r}_i = \sum_{a=1}^{L} \frac{\partial \vec{r}_i}{\partial q_a} dq_a \tag{3.61}$$

Note that **if the constraints do not depend on time**, then $\delta\vec{r}_i = d\vec{r}_i$. This course does not deal with time-dependent constraints, nor will we make a special distinction between possible and virtual displacements. The one exception will be in Chapter 9, where we will consider time-dependent constraints and then $\delta\vec{r}_i \neq d\vec{r}_i$.

Reactions

The particles move according to the constraints, which exert forces on the particles. These are the so-called ***reaction forces*** $\vec{R}_i$.

If, in addition to $\vec{R}_i$, the force $\vec{F}_i$ (generally known) acts on each i particle in the system, we can write the equation of motion (Newton's second law)

$$\vec{F}_i + \vec{R}_i - m_i\vec{a}_i = 0 \quad i = 1 \dots N \tag{3.62}$$

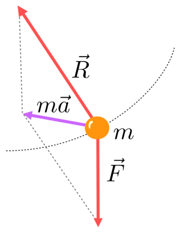


Fig. 3.28: In the case of the pendulum, the $\vec{R}_i$ reaction force is the one exerted by the rod

These equations have to be completed with the constraints (3.56) or (3.57).

The problem we find here is that the constraints are generally known, but their reactions $\vec{R}_i$ are not. As we have mentioned, the reaction forces $\vec{R}_i$ are the forces exerted by the constraints, such that the particles move according to their orders and, of course, also according to Newton's laws (3.62). In equation (3.62), $\vec{R}_i$ are unknown. Only one part of $\vec{r}_i(t)$ is unknown since, as the constraints are known, we will know part of the trajectory of the particles. For example, in the case of the pendulum (see Figure 3.28) restricted to the vertical plane, we already know where the particle passes. We just need to know its speed or, more specifically, we only need to know $\theta(t)$.

In the case of a particle passing through the wire (see Figure 3.29), we also know the trajectory. If the wire is frictionless, we can say that the reaction $\vec{R}$ is normal to it.

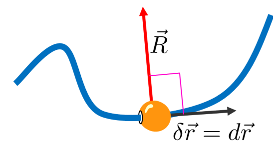


Fig. 3.29: In the absence of friction, the reaction of the wire is normal to it

Ideal reactions

A very interesting property that we observe in the cases represented in Figures 3.28 and 3.29 is that the reaction is normal to the displacement. In the case represented in Figure 3.30, the reaction is normal to the virtual displacement, $\vec{R} \cdot \delta\vec{r} = 0$, but not to the possible displacement, $\vec{R} \cdot d\vec{r} \neq 0$.

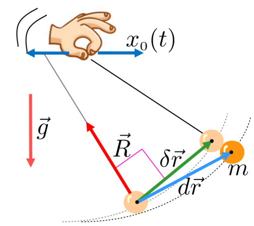


Fig. 3.30: The R tension is normal to the $\delta\vec{r}$ virtual displacement but not to the possible displacement $d\vec{r}$

The fact that the reaction is normal to the displacement means that the work of the force is zero when the system performs the displacement. In fact, what interests us is not that the reactions do not work individually. Although they could work separately, it is enough for us that they do not work together. Driven by this interest, we define the ideal reaction sets below.

➔ **Ideal reaction set**. A set of $\vec{R}_i \quad i = 1 \dots S$ reactions is called a ***set of ideal reactions*** if they fulfil

$$\sum_{i=1}^{S} \vec{R}_i \cdot \delta\vec{r}_i = 0 \tag{3.63}$$

This important property is satisfied in many situations. In general terms, reactions are ideal in the absence of dissipative friction. Let us see a couple of cases.

Problem 3.8.1. Prove that the reaction forces in a frictionless pivot (also called *pivot point*) are ideal.

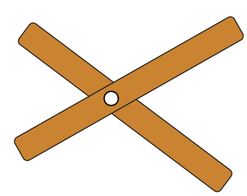
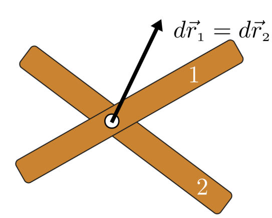


Figure for Problem 3.8.1

Solution

The point where the pivot is located is occupied by one particle from body 1 at position $\vec{r}_1$ and another particle from the body 2 at position $\vec{r}_2$. As a result of the constraint, they move the same, $\vec{r}_1 = \vec{r}_2$. If we differentiate we get $d\vec{r}_1 = d\vec{r}_2$, which can be written as $d\vec{r}_1 - d\vec{r}_2 = 0$.



Solution to Problem 3.8.1

The reaction forces $\vec{R}_1$ and $\vec{R}_2$ fulfil the law of action-reaction $\vec{R}_2 = -\vec{R}_1$. If now we calculate the work

$$\vec{R}_1 \cdot d\vec{r}_1 + \vec{R}_2 \cdot d\vec{r}_2 = \vec{R}_1 \cdot (d\vec{r}_1 - d\vec{r}_2) = 0$$

then the set of forces $\vec{R}_1$ and $\vec{R}_2$ becomes a set of ideal reactions. ■

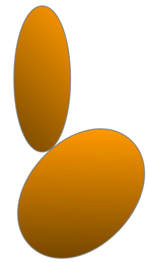


Figure for Problem 3.8.2

Problem 3.8.2. Prove that the reaction forces with frictionless contact between two moving solids are ideal.

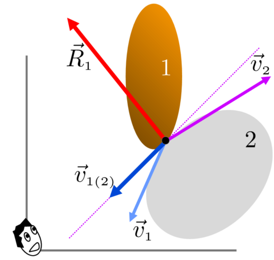
Solution

The bodies have velocities $\vec{v}_1$ and $\vec{v}_2$ with respect to an observer at rest. At any instant, body 1 moves at a velocity $\vec{v}_{1(2)}$ relative to body 2 and *in the direction of the contact plane*. The reaction forces $\vec{R}_2 = -\vec{R}_1$ are normal to this plane and, therefore, they are normal to the relative velocity, that is to say, $\vec{R}_1 \cdot \vec{v}_{1(2)} = 0$.

We can write

$$0 = \vec{R}_1 \cdot (\vec{v}_1 - \vec{v}_2) = \frac{1}{dt}\vec{R}_1 \cdot (d\vec{r}_1 - d\vec{r}_2) =$$
$$\frac{1}{dt}\left(\vec{R}_1 \cdot d\vec{r}_1 - \vec{R}_1 \cdot d\vec{r}_2\right) = \frac{1}{dt}\left(\vec{R}_1 \cdot d\vec{r}_1 + \vec{R}_2 \cdot d\vec{r}_2\right)$$

where $d\vec{r}_1$ and $d\vec{r}_2$ are the possible displacements. Therefore, the contact reactions are a set of ideal reactions. ■



Solution to Problem 3.8.2

3.9 The general equation of dynamics also known as d'Alembert's principle

We want to write some equations of motion in which the constraints appear with out the reaction forces. That is, we want to rewrite Newton's second law while taking into account that the constraints are part of the data while their reactions are unknown.

The set of forces acting on the system is classified into two subsets: the reactions $\vec{R}_i$ and the other forces, $\vec{F}_i$, which we will call ***directly applied forces***.

➔ ***General equation of dynamics***. *If all the reactions of the system are ideal, Newton's equations of motion are equivalent to*

$$\sum_{i=1}^{N}\left(\vec{F}_i - m_i\vec{a}_i\right) \cdot \delta\vec{r}_i = 0 \qquad (3.64)$$

We note that in the equations extracted from (3.64):

➔ **1)** Reaction forces do not appear.

➔ **2)** We obtain as many independent equations as we do degrees of freedom.

The first version of the general equation of dynamics came from Jean le Rond d'Alembert and was given the name of d'Alembert's principle, stated in the following way:

Any position of a moving system is in equilibrium if inertial forces are added to external forces.

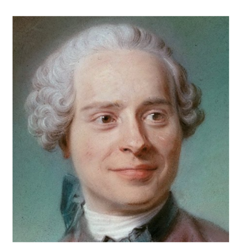


Fig. 3.31: Jean le Rond D'Alembert (1717-1783) was a French mathematician and philosopher

This statement will be later understood in the context of statics and in relation to the principle of virtual works. However, we will not use this interpretation.

The proof of the general equation of dynamics (3.64) is not difficult from a mathematical point of view, although it may be conceptually surprising.

Proof. We want to prove that

$$\sum_{i=1}^{N} \vec{R}_i \cdot \delta\vec{r}_i = 0 \tag{3.65}$$

and

$$\vec{F}_i + \vec{R}_i = m_i\vec{a}_i \tag{3.66}$$

are equivalent to

$$\sum_{i=1}^{N} \left(\vec{F}_i - m_i\vec{a}_i\right) \cdot \delta\vec{r}_i = 0 \tag{3.67}$$

Proof (3.65, 3.66) $\Rightarrow$ (3.67): If we multiply (3.66) by $\delta\vec{r}_i$ and sum over index i, we obtain

$$0 = \sum_{i=1}^{N} \left(\vec{F}_i + \vec{R}_i - m_i\vec{a}\right) \cdot \delta\vec{r}_i$$

Thus, taking into account (3.66), we obtain (3.67).

Proof (3.67) $\Rightarrow$ (3.65, 3.66): We now observe that the reactions are not determined. They have to enforce the system's motion through the constraints. Thus, they are determined as $\vec{R}_i = m_i\vec{a}_i - \vec{F}_i$ and, obviously, (3.66) is fulfilled. If we multiply (3.66) by $\delta\vec{r}_i$ and sum over index i, we obtain $\sum_{i=1}^{N} \left(\vec{F}_i + \vec{R}_i - m_i\vec{a}_i\right) \cdot \delta\vec{r}_i = 0$. Now, taking into account (3.67), we obtain (3.65). ■

3.10 Conservative system with constraints. Energy conservation

Here we consider a conservative system with constraints: a particle system with time-independent constraints that all have ideal reactions $\vec{R}_i$ and, furthemore, the other forces $\vec{F}_i$ (internal or external) are conservative. This means that the system will have the potential energy function U:

$$U = -\int \sum_{i=1}^{N} \vec{F}_i \cdot d\vec{r}_i \tag{3.68}$$

➔ ***Conservation of mechanical energy theorem***. *If we define the mechanical energy in a constrained conservative system of N particles as*

$$E = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2 + U \tag{3.69}$$

where U is defined according to (3.68), then the numerical value is maintained constant over time.

$$\frac{dE}{dt} = 0 \tag{3.70}$$

Proof. We apply the general equation of dynamics by taking as possible displacements the actual displacements of the system (the displacements that fulfil the general equation of dynamics) and divide by elapsed time, dt:

$$\sum_{i=1}^{N} \left(\vec{F}_i - m_i\vec{a}_i\right) \cdot \frac{d\vec{r}_i}{dt} = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{d\vec{r}_i}{dt} - \sum_{i=1}^{N} m_i\vec{a}_i \cdot \vec{v}_i \tag{3.71}$$

$$= -\left(\frac{dU}{dt} + \frac{d}{dt}\frac{1}{2}\sum_{i=1}^{N} m_i v_i^2\right) = 0 \tag{3.72}$$

■

It is usually said that *mechanical energy is conserved*. Energy conservation can also be written using the integrated expression between two instants t_{ini} and t_{fi}

$$E_{\text{ini}} = E_{\text{fi}} \tag{3.73}$$

➔ We must remember that the constraints must be time-independent in order for the energy to be conserved. It is not enough that the directly applied forces are conservative and the reactions are ideal. The virtual displacements and possible displacements must also coincide: $\delta\vec{r}_i = d\vec{r}_i$. Throughout the book, we will consider primarily time-independent constraints, and only Chapter 9 will deal with time-dependent constraints.

➔ In the case of conservative systems with one degree of freedom, this result is sufficient for finding the temporal trajectory of the system.

It should be noted that $d\vec{r}_i$ from (3.68) are the possible particle displacements to which the forces are applied, but they do not need to coincide with the point displacements in the space where the forces are applied. We can see this in the following example.

Problem 3.10.1. In the figure, spring is relaxed when the lower face A of a prism with section S and mass m, which can slide along a fixed track, is at the level of a liquid with density ρ. If we let it go from the described position, what will be the depth y what A will reach?

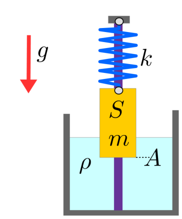


Figure for Problem 3.10.1

Note: The container is large enough so that the change in the liquid level is negligible.

Solution

It is rigid body that can be moved only by translation. We take the y-axis in the direction of $\vec{g}$ with the origin at the fixed level of the liquid. Under these conditions, the point A coordinate is y. When $y=0$, the spring exerts no force and Archimedes' buoyant force is null. The possible infinitesimal displacement of any particle of the body is $d\vec{r} = dy\hat{\jmath}$.

The system is conservative and, as a result, we can apply energy conservation between the initial and final situations. To be able to write the different potential energy terms, we will consider point A to be sunk to y:

U_g: the force is $\vec{P} = mg\hat{\jmath}$ and the associated potential energy $U_g = -\int (mg\hat{\jmath}) \cdot (dy\,\hat{\jmath}) = -mgy$.

U_k: the force exerted by the spring is $\vec{F}_k = -ky\hat{\jmath}$ and the associated potential energy, $U_k = -\int (-ky\hat{\jmath}) \cdot (dy\,\hat{\jmath}) = \frac{1}{2}ky^2$.

U_E: Arhimedes' buoyancy (see Section 4.3) is $\vec{E} = -\rho g S y\,\hat{\jmath}$ and the associated potential energy, $U_E = -\int (-\rho g S y\hat{\jmath}) \cdot (dy\,\hat{\jmath}) = \frac{1}{2}\rho g S y^2$.

Let us note that the displacement is always that of the particle in the body where the corresponding force is applied. In particular, we do not need to know, at each instant, which is the point where Archimedes' buoyant force E is applied, which we will study in the next chapter. We will see that when A is at y, the buoyancy E is applied to $y/2$ and, therefore the displacement of the E application point is $dy/2$. However, we now need the possible displacement dy of the **particle of the body** where E is applied, although this particle may be different during the process.

The mechanical energy of the system is

$$E(y,\dot{y}) = \frac{1}{2}m\dot{y}^2 - mgy + \frac{1}{2}ky^2 + \frac{1}{2}\rho g S y^2 = \frac{1}{2}m\dot{y}^2 - mgy + \frac{1}{2}(k+\rho g S)\,y^2$$

applying its conservation, we have $E(0,0) = E(y,0) \Rightarrow$ $y = \dfrac{2mg}{k+\rho g S}$ ■

Problem 3.10.2. A particle tied to a rope of negligible mass and that at all times remains tense, oscillates in a vertical plane. Find the motion equation using energy conservation.

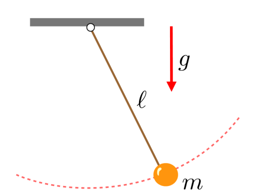


Figure for Problem 3.10.2

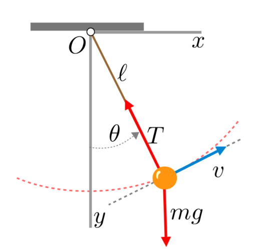
Solution

We observe that the tension T of the rope is always normal to the possible particle displacement. This is an ideal reaction. Aside from the reactions, we have the gravity force, which is conservative. We can express the constraint in the parametric form as

$$x = \ell\,\sin\theta$$
$$y = \ell\,\cos\theta$$

The energy system is $E = \frac{1}{2}mv^2 - mg\ell\cos\theta$, where $v = \sqrt{\dot{x}^2 + \dot{y}^2}$ which, in terms of θ, is $v = \ell\dot{\theta}$. Thus, we obtain

$$E = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta$$



Solution to Problem 3.10.2

and applying energy conservation, we have

$$0 = \frac{dE}{dt} = m\ell^2\,\dot\theta\,\ddot\theta + mg\ell\sin\theta\,\dot\theta$$

Taking into account that we look for solutions allowed by the constraints $\dot\theta \neq 0$, we obtain the motion equation

$$\ddot\theta = -\frac{g}{\ell}\sin\theta \qquad \blacksquare$$

Problem 3.10.3. In the figure, nail C is at a vertical distance d from where a rope attached to a ball of mass m is fixed. By letting the ball go so that it can make a complete revolution in a circle centred on the nail, prove that it is necessary that $d \geq 0.6L$.

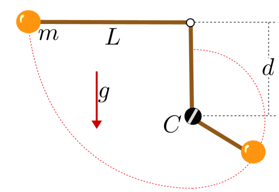


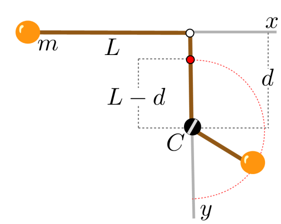
Figure for Problem 3.10.3

Solution

It is a constrained conservative system with ideal constraints. We will use energy conservation $E_{\text{ini}} = E_{\text{fi}}$ between the ini position, with $v = 0$ and $y = 0 \Rightarrow E_{\text{ini}} = 0$, and the fi position, with y and $v \Rightarrow E_{\text{fi}} = \frac{1}{2}mv^2 - mgy$.

At the red point in the figure, we have $y = d - (L - d)$. Substituting y in $E_{\text{ini}} = E_{\text{fi}}$ and isolating the speed, we have

$$v = \sqrt{2g(2d - L)} \tag{1}$$



Solution to Problem 3.10.3

At the same time, this speed must be sufficient so that the rope does not slacken. Thus, at the point where it has the least speed, the tension T of the rope must keep it stretched, that is, $T \geq 0$. By writing the motion equation in the tension direction and taking into account that the normal acceleration is $a_n = \frac{v^2}{L-d}$,

$$mg + T = m\frac{v^2}{L - d}$$

Isolating v and imposing that $T \geq 0$, we have

$$v \geq \sqrt{g(L - d)} \tag{2}$$

Combining (1) and (2):

$$\sqrt{2g(2d - L)} \geq \sqrt{g(L - d)} \Rightarrow 4d - 2L \geq L - d \Rightarrow d \geq \frac{3}{5}L \qquad \blacksquare$$

Problem 3.10.4. The blocks in the figure are released from rest. There is neither friction on the horizontal ground nor in the pulleys nor in the rope, which are of negligible mass. Determine the acceleration of the blocks and the tension in the rope.

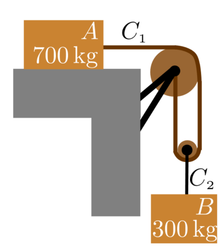
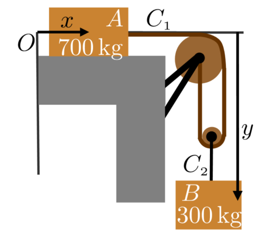


Figure for Problem 3.10.4

Solution

It is a constrained conservative system. Given that when we make a possible displacement dx, once the section of rope that advances dx past the pulley must be divided between two rope sections, thus giving us $dy = \frac{1}{2}dx$ and, dividing by dt, $\dot{y} = \frac{1}{2}\dot{x}$.



Solution to Problem 3.10.4

We can write the energy of the system, directly using the numerical values of the masses, as

$$E = \frac{1}{2}700\,\dot{x}^2 + \frac{1}{2}300\,\dot{y}^2 - 300 \times 9.81\,y = 387.5\,\dot{x}^2 - 2943y$$

Deriving with respect to time, we have

$$\dot{E} = 2 \times 387.5\dot{x}\,\ddot{x} - 2943\dot{y}$$

Substituting the constraint $\dot{y} = \frac{1}{2}\dot{x}$ and imposing energy conservation give us

$$(775\,\ddot{x} - 1471.5)\dot{x} = 0$$

Taking into account that we are looking for solutions compatible with the constraints, $\dot{x} \neq 0$, we obtain

$$\ddot{x} = 1.8987\,\mathrm{m/s^2} \qquad \ddot{y} = 0.94935\,\mathrm{m/s^2}$$

Now applying Newton's second law to each block:

$$T_1 = 700\,\ddot{x} \Rightarrow T_1 = 1329.1\,\mathrm{N}$$

$$300 \times 9.81 - T_2 = 300\,\ddot{y} \Rightarrow T_2 = 2658.2\,\mathrm{N}$$

■

Problem 3.10.5. The 6 kg block B is left to slide down on the 15 kg wedge A resting on the frictionless horizontal ground. Calculate the accelerations of A and B.

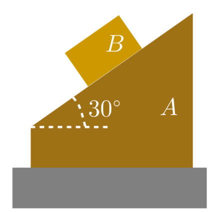


Figure for Problem 3.10.5

Solution

There is no friction and there are no external forces on the horizontal ground direction. The conservation of momentum imposes

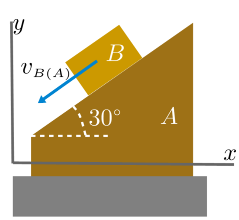
$$m_A\dot{x}_A + m_B\dot{x}_B = 0 \Rightarrow$$

$$\dot{x}_A = -\frac{m_B}{m_A}\dot{x}_B = -\frac{2}{5}\dot{x}_B \qquad (1)$$

In the figure, we see that the velocity of B relative to A has an inclination of 30° (We have to take into account that the inclined plane moves!).

$$\vec{v}_{B(A)} = (\dot{x}_B - \dot{x}_A, \dot{y}_B - \dot{y}_A) \quad \Rightarrow \quad \frac{\dot{y}_B}{(\dot{x}_B - \dot{x}_A)} = \tan 30^\circ \quad \Rightarrow$$

$$\dot{y}_B = \frac{7}{5\sqrt{3}}\dot{x}_B \qquad (21)$$



Solution to Problem 3.10.5

Now we use energy conservation. To do this, we write the energy as a function of $\dot{x}_B$ and y_B, derive it and equal to zero

$$E = \tfrac{1}{2} m_A \dot{x}_A^2 + \tfrac{1}{2} m_B (\dot{x}_B^2 + \dot{y}_B^2) + m_B g y_B = 6.16\, \dot{x}_B^2 + 58.86 y_B$$

$$\dot{E} = 0 = 12.32\, \dot{x}_B \ddot{x}_B + 47.576 \dot{x}_B \Rightarrow \boxed{\ddot{x}_B = -\frac{47.576}{12.32} = -3.86\,\text{m/s}^2}$$

Substituting in (1) and (2) derived with respect to time, it follows that

$$\boxed{\ddot{y}_B = -3.12\,\text{m/s}^2} \qquad \boxed{\ddot{x}_A = 1.545\,\text{m/s}^2}$$

■

Problem 3.10.6. A mass $m = 0.5\,\text{kg}$ slides without friction in a vertical plane along a wire ($d = 0.8\,\text{m}$). The spring has a natural length of $\ell = 25\,\text{cm}$ and $k = 600\,\text{N/m}$. If the mass is released with no initial velocity when $b = 30\,\text{cm}$, determine:

a) The velocity when it reaches C.

b) The velocity when it reaches B.

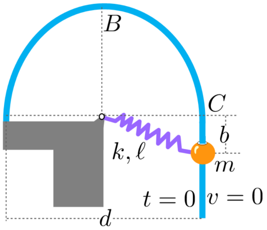


Figure for Problem 3.10.6

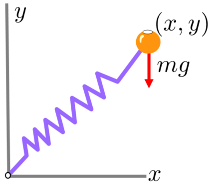
Solution

It is a conservative system with an ideal constraint. The conservative forces are the weight and the force exerted by the spring. The energy as a function of x, y and v can be written as

$$E(x, y, v) = \frac{1}{2} m v^2 + m g y + \frac{1}{2} k \left(\ell - \sqrt{x^2 + y^2}\right)^2$$

where x and y have to be on the wire which is the constraint. Let us impose energy conservation between the initial point and the points indicated in each part. The energy at the initial point is

$$E = \frac{1}{2} m 0^2 + m g(-0.3) + \frac{1}{2} k \left(\ell - \sqrt{0.4^2 + (-0.3)^2}\right)^2 = 17.2785\,\text{J}$$



Solution to Problem 3.10.6

a)

$$\frac{1}{2} m\, v_C^2 + m g\ 0 + \frac{1}{2}\ k\ \ (\ell - 0.4)^2 = 17.2785 \Rightarrow \boxed{v_C = 6.48953\,\text{m/s}}$$

b)

$$\frac{1}{2} m\, v_B^2 + m g\ 0.4 + \frac{1}{2}\ k\ \ (\ell - 0.4)^2 = 17.2785 \Rightarrow \boxed{v_B = 5.85372\,\text{m/s}}$$

■

3.11 Rigid body

A rigid body is a system of i particles of $\vec{r}_i$ positions with geometrical constraints

$$(\vec{r}_i - \vec{r}_j)^2 = \text{ct} \tag{3.74}$$

Ideal reactions

➔ The cohesive forces of a rigid body form a set of ideal reactions.

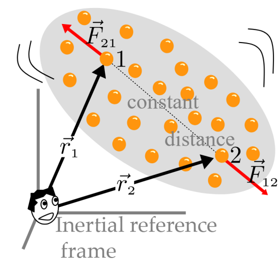


Fig. 3.32: In a rigid body, the internal forces are the reaction forces of the constraints that keep the distance between particles constant

Proof. We will analyse any two of the many particles in the body, 1 and 2, with position vectors $\vec{r}_1$ and $\vec{r}_2$, respectively (see Figure 3.32). The cohesive forces of the body (internal forces) are the constraint reactions. The forces $\vec{F}_{12}$ and $\vec{F}_{21}$ will satisfy Newton's third law:

$$\vec{F}_{21} = -\vec{F}_{12} \quad ; \quad \vec{F}_{12} \propto (\vec{r}_2 - \vec{r}_1) \tag{3.75}$$

The possible (= virtual) $d\vec{r}_1$ and $d\vec{r}_2$ displacements will fulfil

$$0 = d(\vec{r}_2 - \vec{r}_1)^2 = 2(\vec{r}_2 - \vec{r}_1) \cdot (d\vec{r}_2 - d\vec{r}_1) = 0 \tag{3.76}$$

The condition for ideal constraints for these two particles is $\vec{F}_{21} \cdot d\vec{r}_1 + \vec{F}_{12} \cdot d\vec{r}_2 = 0$ and that is what we must prove. Taking into account (3.75) and (3.76), we have

$$\begin{aligned} &\vec{F}_{21} \cdot d\vec{r}_1 + \vec{F}_{12} \cdot d\vec{r}_2 = -\vec{F}_{12} \cdot d\vec{r}_1 + \vec{F}_{12} \cdot d\vec{r}_2 = \\ &\vec{F}_{12} \cdot (d\vec{r}_2 - d\vec{r}_1) \;\propto\; (\vec{r}_2 - \vec{r}_1) \cdot (d\vec{r}_2 - d\vec{r}_1) = 0 \end{aligned}$$

We can extend this result to all the particle pairs in the body and, thus, the set of cohesive forces in a rigid body is ideal. ■

Possible displacements

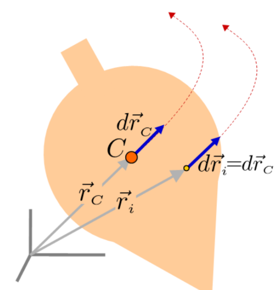


Fig. 3.33: Translation without rotation

From a body, we take a reference point C, of position $\vec{r}_C$ (see Figure 3.33). The position of any other point in the body can be written as $\vec{r}_i = \vec{r}_C + \vec{r}_{i(C)}$. We can thus decompose the possible displacements $d\vec{r}_i$ as

$$d\vec{r}_i = d\vec{r}_C + d\vec{r}_{i(C)} \tag{3.77}$$

If we substitute $\vec{r}_i = \vec{r}_C + \vec{r}_{i(C)}$ into condition (3.74), the result does not depend on $\vec{r}_C$. Thus, any $d\vec{r}_C$ is possible. When the possible displacement has the form

$$d\vec{r}_i = d\vec{r}_C \tag{3.78}$$

we say that it is a **translation**.

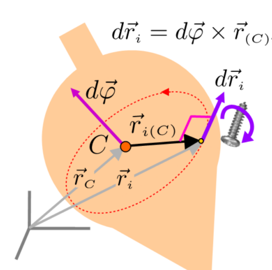


Fig. 3.34: Rotation without translation

It remains to be seen what restrictions are imposed by (3.74) on $d\vec{r}_{i(C)}$. The condition is

$$d\left(\vec{r}_i - \vec{r}_j\right)^2 = 2(\vec{r}_{i(C)} - \vec{r}_{j(C)}) \cdot (d\vec{r}_{i(C)} - d\vec{r}_{j(C)}) = 0 \tag{3.79}$$

The solution to these equations gives us the expression of the possible displacements with respect to point C, $d\vec{r}_{j(C)}$, which we denote as ***rotations with respect to*** C (see Figure 3.34).

➔ *The general expression for a possible rotational displacement with respect to C is*

$$d\vec{r}_{i(C)} = d\vec{\varphi} \times \vec{r}_{i(C)} \tag{3.80}$$

where $d\vec{\varphi}$ is any infinitesimal vector (three degrees of freedom).

The vector $d\vec{\varphi}$ defines the infinitesimal rotation of the body as a whole. The rotation is not around a point, but is understood as the body's change of orientation in space. The direction $d\vec{\varphi}$ represents the **instantaneous axis of rotation**, together with the corkscrew rule, and the modulus represents the (infinitesimal) angle of this rotation.

It should be noted that the d symbol used in $d\vec{\varphi}$ cannot mean, in general, differentiation. We say that, in general, $d\vec{\varphi}$ is not integrable; we cannot find $\vec{\varphi}$ for any rotation (because it does not exist!). We can do so if, for example, the direction of $d\vec{\varphi}$ is constant.

Proof. We want to show that (3.80) is a solution to (3.79). If we take $\vec{r}_j = \vec{r}_C \Rightarrow \vec{r}_{j(C)} = \vec{r}_{C(C)} = 0$, from (3.79) we get

$$\vec{r}_{i(C)} \cdot d\vec{r}_{i(C)} = 0 \tag{3.81}$$

if we now use (3.81), calculating the products in (3.79), we get

$$\vec{r}_{i(C)} \cdot d\vec{r}_{j(C)} + \vec{r}_{j(C)} \cdot d\vec{r}_{i(C)} = 0 \tag{3.82}$$

Thus, (3.81) and (3.82) are the conditions to be satisfied by the possible displacements $d\vec{r}_{i(C)}$. Equation (3.81) tells us that $d\vec{r}_{i(C)} \perp \vec{r}_{i(C)}$ and, therefore, we can express $d\vec{r}_{i(C)}$ as $d\vec{r}_{i(C)} = d\vec{\varphi}_i \times \vec{r}_{i(C)}$. Substituting in (3.82), we get

$$\vec{r}_{i(C)} \cdot (d\vec{\varphi}_j \times \vec{r}_{j(C)}) + \vec{r}_{j(C)} \cdot (d\vec{\varphi}_i \times \vec{r}_{i(C)}) = 0$$

which, taking into account the vector property $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$, can be rewritten as

$$(\vec{r}_{i(C)} \times \vec{r}_{j(C)}) \cdot (d\vec{\varphi}_j - d\vec{\varphi}_i) = 0$$

Since this has to happen for every pair of i and j particles in the body $\Rightarrow d\vec{\varphi}_j = d\vec{\varphi}_i = d\vec{\varphi}$ ■

A general possible (infinitesimal) displacement is the composition of a translation and a rotation, i.e.

$$d\vec{r}_i = d\vec{r}_C + d\vec{\varphi} \times \vec{r}_{i(C)} \tag{3.83}$$

3.12 Topics in rigid body kinematics

If each member in relation (3.83) is divided by dt, we obtain the expression of the instantaneous velocity of any particle of the body, $\vec{v}_i = \frac{d\vec{r}_i}{dt}$, which is a function of the velocity $\vec{v}_C = \frac{d\vec{r}_C}{dt}$ of reference point C and of the **angular velocity**, $\vec{\omega}$, $\vec{\omega} = \frac{d\vec{\varphi}}{dt}$ (see Figure 3.35):

$$\vec{v}_i = \vec{v}_C + \vec{\omega} \times \vec{r}_{i(C)} \tag{3.84}$$

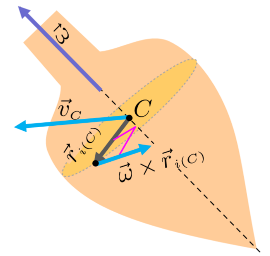


Fig. 3.35: Instantaneous velocity of the particle and of a rigid body in rotation

The angular velocity is the body's instantaneous velocity of rotation according to the instantaneous axis defined by the line of action of $\vec{\omega}$, direction of rotation according to the corkscrew rule and modulus ω. Note that the different points in the body can have different velocities $\vec{v}_i$, but the body has **one** angular velocity $\vec{\omega}$.

The application of relation (3.84) to any two particles i and j of the body can be written as follows:

$$\begin{cases} \vec{v}_i = \vec{v}_C + \vec{\omega} \times \vec{r}_{i(C)} \\ \vec{v}_j = \vec{v}_C + \vec{\omega} \times \vec{r}_{j(C)} \end{cases}$$

If we subtract these two equations member by member, it follows that

$$\vec{v}_i - \vec{v}_j = \vec{\omega} \times (\vec{r}_i - \vec{r}_j) \tag{3.85}$$

since $\vec{r}_{i(C)} - \vec{r}_{j(C)} = \vec{r}_i - \vec{r}_j$.

According to the vector product properties, it is evident that

$$(\vec{v}_i - \vec{v}_j) \cdot (\vec{r}_i - \vec{r}_j) = 0 \tag{3.86}$$

$$(\vec{v}_i - \vec{v}_j) \cdot \vec{\omega} = 0 \tag{3.87}$$

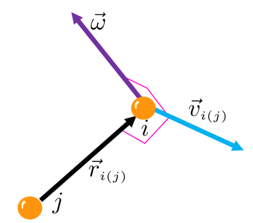


Fig. 3.36: Relative position and velocity of two particles of a rotating body

from which it follows that the relative velocity vector between two particles is, at all times, perpendicular to the vectors' relative position and angular velocity (see Figure 3.36).

The point i where $\vec{v}_i = 0$ is called the ***instantaneous rotation centre*** (IRC). See Figure 3.37. According to (3.86) $\vec{r}_{IRC(j)}$ fulfils

$$\vec{v}_j \cdot \vec{r}_{IRC(j)} = 0 \tag{3.88}$$

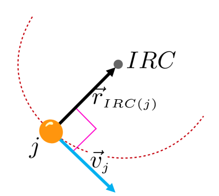


Fig. 3.37: Instantaneous rotation centre (IRC)

In general, IRC cannot be identified with any point rigidly attached to the body.

Problem 3.12.1. A ladder $\overline{AB}$, 3 m long, slides down a wall and across a floor. When θ is 30°, the lower end of the ladder moves to the right at a velocity of 2.0 m/s. Determine the velocity of the upper end and the angular velocity at this instant.

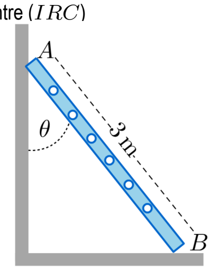
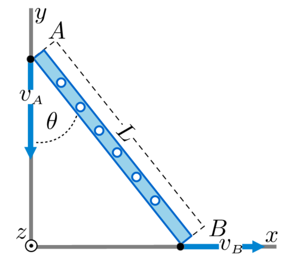


Figure for Problem 3.12.1

Solution

If equation (3.85) is applied to points A and B on the ladder, it follows that

$$\vec{v}_A - \vec{v}_B = \vec{\omega} \times (\vec{r}_A - \vec{r}_B) \tag{1}$$



Solution to Problem 3.12.1

Based on reference frame shown in the figure and taking into account that $\vec{r}_A - \vec{r}_B = (-L\sin\theta, L\cos\theta, 0)$, that $\vec{v}_A = (0, -v_A, 0)$, $\vec{v}_B = (v_B, 0, 0)$ and $\vec{\omega} = (0, 0, \omega)$, (1) becomes

$$(0, -v_A, 0) - (v_B, 0, 0) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \omega \\ -L\sin\theta & L\cos\theta & 0 \end{vmatrix}$$

Solving the determinant give us

$$-v_B\hat{\imath} - v_A\hat{\jmath} = -L\omega\cos\theta\,\hat{\imath} - L\omega\sin\theta\,\hat{\jmath}$$

and equating the corresponding components, it follows that

$$\begin{aligned} v_B &= \omega L\cos\theta \\ v_A &= \omega L\sin\theta \end{aligned}$$

Substituting the data provided by the statement, $L = 3$ m, $v_B = 2$ m/s and $\theta = 30°$, we finally obtain

$$\boxed{v_A = 1.155\,\text{m/s}}\,,\ \boxed{\omega = 0.77\,\text{rad/s}}$$

■

Problem 3.12.2. At the instant shown in the figure, slide A is moving to the right at a velocity of $v_A = 3\,\text{m/s}$. Find the angular velocity ω of the bar $\overline{AB}$ and the velocity v_B of slide B.

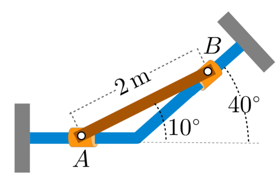


Figure for Problem 3.12.2

Solution

As in Problem 3.12.1, applying equation (3.85) to points A and B of the bar joining both slides allows us to write

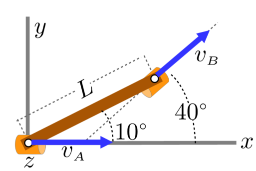
$$\vec{v}_A - \vec{v}_B = \vec{\omega} \times (\vec{r}_A - \vec{r}_B)$$

Taking the reference frame shown in the solution figure and considering that

$$\vec{r}_A - \vec{r}_B = -L\cos 10°\,\hat{\imath} - L\sin 10°\,\hat{\jmath}$$

and that $\vec{v}_A = v_A\hat{\imath}$, $\vec{v}_B = v_B\cos 40°\hat{\imath} + v_B\sin 40°\hat{\imath}$ and $\vec{\omega} = \omega\hat{k}$, this last expression can be written as follows:

$$(v_A, 0, 0) - (v_B\cos 40^0, v_B\sin 40^0, 0) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \omega \\ -L\cos 10^0 & -L\sin 10^0 & 0 \end{vmatrix}$$



Solution to Problem 3.12.2

Solving the determinant, we have

$$v_A\hat{\imath} - v_B\cos 40°\hat{\imath} - v_B\sin 40°\hat{\jmath} = -L\omega\sin 10°\hat{\imath} - L\omega\cos 10°\hat{\jmath}$$

and, equalizing the corresponding components, we arrive at

$$\begin{aligned} v_A - v_B\cos 40° &= \omega L\sin 10° \\ v_B\sin 40° &= \omega L\cos 10° \end{aligned}$$

Based on the data provided by the problem statement, the solution to this system allows us to obtain v_B and ω:

$$\boxed{v_B = 3.41\,\text{m/s}}\,,\ \boxed{\omega = 1.113\,\text{rad/s}}$$

■

Non-slip rolling

Consider a rigid body that moves without losing contact with the flat ground (see Figure 3.38). Point B of the body in contact with the ground changes with time and its velocity relative to the ground, $\vec{v}_B - \vec{v}_{\text{ground}}$, may or may not be different from zero. If it is non-zero the body is said to **roll and slide**, while if it is zero the body is said to **roll without sliding** (or it **rolls**).

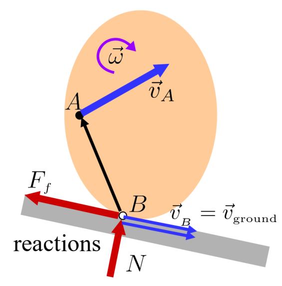


Fig. 3.38: Rigid body in motion on a fixed flat surface

Applying equation (3.84) to points A and B of the body (see Figure 3.38), it follows that

$$\vec{v}_A = \vec{v}_B + \vec{\omega} \times (\vec{r}_A - \vec{r}_B)$$

If the body rolls without sliding, $\vec{v}_B = \vec{v}_{\text{ground}}$ and the above expression can be written as

$$\vec{v}_A = \vec{v}_{\text{ground}} + \vec{\omega} \times (\vec{r}_A - \vec{r}_B)$$

If $\vec{v}_{\text{ground}} = 0$, then $\vec{v}_A$ is perpendicular to the straight line joining point A with point B, which is where the body has instantaneous contact with the ground. If we take into account that $\vec{\omega}$ and $\vec{r}_A - \vec{r}_B$ are perpendicular vectors, it follows that

$$v_A = \omega \overline{AB} \tag{3.89}$$

from which it also follows that v_A is directly proportional to the distance $\overline{AB}$. In this way, we obtain a distribution of instantaneous velocity vectors in the body, as shown in Figure 3.39. Point B is, in this case, the instantaneous centre of rotation (ICR), defined in (3.88). In many cases, the body will have a circular section of radius R. If equation (3.89) is now applied to points B and C (centre) of the body, $\overline{BC} = R$, we obtain

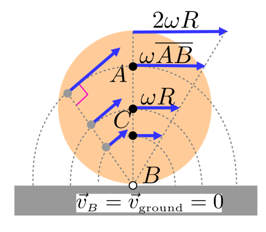


Fig. 3.39: Velocity distribution in the plane of motion of a bopdy rolling without sliding on a floor

$$v_C = \omega R$$

Note 1: When the body rolls without sliding, the point in contact with the ground (point B in Figures 3.38 and 3.39) instantaneously has zero relative velocity, which produces ideal reactions of the ground on the body, and these reactions are applied at the point of contact (Normal N and friction F_f in Figure 3.38). This is especially important with regard to friction, since the normal is already ideal in the case of sliding.

Note 2: If the body rolls without sliding, then $F_f \leq \mu_s N$ while if it rolls and slides, then $F_f = \mu_k N$ where μ_s and μ_k are, respectively, the static and dynamic friction coefficients between the body and the ground.

3.13 Equations of motion for a rigid body

The general equation of dynamics for the rigid body under external forces $\vec{F}_i$ is obtained by taking as virtual displacements $\delta\vec{r}_i = d\vec{r}_C + d\vec{\varphi} \times \vec{r}_{i(C)}$:

$$\sum_{i=1}^{N} \left(\vec{F}_i - m_i\vec{a}_i\right) \cdot d\vec{r}_C + \sum_{i=1}^{N} \left(\vec{F}_i - m_i\vec{a}_i\right) \cdot (d\vec{\varphi} \times \vec{r}_{(C)\,i}) = 0 \tag{3.90}$$

Recall that we can make $d\vec{r}_C$ and $d\vec{\varphi}$ to have any *infinitesimal value*, since they are independent of each other. Thus, each factor of $d\vec{\varphi}$ and $d\vec{r}_C$ in (3.90) must to be zero. In particular the factor of $d\vec{r}_C$ fullfil $\sum_{i=1}^{N}\left(\vec{F}_i - m_i\vec{a}_i\right) = 0$, an equation which we have already discussed above but which now allows us to state:

➔ *The equation of motion for a rigid body associated with the translations can be written as follows:*

$$\vec{F} = m\,\vec{a}_{CM} = \frac{d\vec{P}}{dt} \tag{3.91}$$

To find the equation of motion associated with the rotations, a little more work is needed.

➔ *The equation of motion for a rigid body associated with the rotations can be written as follows:*

$$\frac{d\vec{L}_{(C)}}{dt} = \vec{M}_{(C)} \tag{3.92}$$

where C is the CM or, if it exists, a fixed point rigidly attached to the body. $\vec{L}_{(C)}$ and $\vec{M}_{(C)}$ are, respectively, the angular momentum of the body and the momentum of the external forces with respect to point C, both calculated in the inertial reference frame.

Proof. As explained above, factor of $d\vec{\varphi}$ in the general equation of dynamics (3.90) must be null. It is not so easy to extract this factor. We start with

$$\sum_{i=1}^{N} \left(\left(\vec{F}_i - m_i\vec{a}_i\right) \cdot (d\vec{\varphi} \times \vec{r}_{i(C)})\right) = 0 \tag{3.93}$$

Taking into account the relationship $\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = -\left(\vec{A} \times \vec{C}\right) \cdot \vec{B}$ with $\vec{A} = \vec{F}_i - m_i\ddot{\vec{r}}_i$, $\vec{B} = d\vec{\varphi}$ and $\vec{C} = \vec{r}_{i(C)}$, we get

$$-\sum_{i}^{N} \left(\left(\vec{F}_i - m_i\ddot{\vec{r}}_i\right) \times \vec{r}_{i(C)}\right) \cdot d\vec{\varphi} = 0 \tag{3.94}$$

which, since $d\vec{\varphi}$ is arbitrary, tells us that

$$\sum_{i}^{N} \left(\left(\vec{F}_i - m_i\ddot{\vec{r}}_i\right) \times \vec{r}_{i(C)}\right) = 0 \tag{3.95}$$

The term $\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \times \vec{r}_{i(C)}$ can be rewritten as

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \times \vec{r}_{i(C)} = \frac{d}{dt} \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \times \vec{r}_{i(C)} - \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \times \dot{\vec{r}}_{i(C)}$$

With $\vec{r}_i = \vec{r}_C + \vec{r}_{i(C)} \Rightarrow \dot{\vec{r}}_i = \dot{\vec{r}}_C + \dot{\vec{r}}_{i(C)}$, and provided we choose the CM as point C or, if it exists, a fixed point rigidly attached to the body ($\dot{\vec{r}}_C = 0$), we have

$$\sum_{i=1}^{N} m_i \dot{\vec{r}}_i \times \dot{\vec{r}}_{i(C)} = \dot{\vec{r}}_C \times \sum_{i=1}^{N} m_i \dot{\vec{r}}_{i(C)} = 0$$

Equation (3.95) finally reads as follows

$$\frac{d}{dt} \sum_{i=1}^{N} \vec{r}_{i(C)} \times m_i \dot{\vec{r}}_i = \sum_{i} \left(\vec{r}_{i(C)} \times \vec{F}_i \right) \tag{3.96}$$

which is nothing other than expression (3.92) made explicit. ■

Equations of motion (3.91) and (3.92) were found earlier in Sections 3.3 and 3.4, although with a slightly different meaning. There, they arose as linear combinations of the equations for each particle, so fulfilling them was necessary for any system of N particles. Moreover, the point for calculating angular moments and torques had to be a fixed point (even if the system was in motion)[2]. Here, they are the result of applying the general equation of dynamics to a rigid body and, therefore, **they are necessary and sufficient for a rigid body**. In other words, if the position and velocity at a given instant and the forces acting on the body are known, its future (and its past) can be predicted. Thus, with respect to the equations of motion in the form of (3.91) and (3.92), we can conclude: If C is a fixed point not rigidly attached to the body, the equations are necessary; if C is the CM or a point rigidly attached to the body, they are necessary and sufficient.

[2] There, we already advanced that the CM could also be used

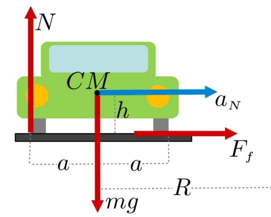
Another remarkable aspect is that, if the moment of the forces acting on the body is zero with respect to CM, the motion for rigid body can be described only by (3.91), by which it can be interpreted as behaving like a particle of the same mass concentrated at the CM point.

Equivalent system of forces

The equations of motion of the body allow defining the sets of forces that will have the same effects on a rigid body. By observing only the motion of the body, we cannot tell whether the cause is one set of forces or the other.

➔ **Equivalent systems of forces**. If two sets of forces applied to a rigid body have equal net force and equal net moment (with respect to the same point), they will cause the same physical effects on the body. We say that they are two equivalent force systems.

Problem 3.13.1. A car of mass m with wheel width $2a$ is travelling at a constant velocity modulus v on a horizontal track, such that its CM at a height h above the track, describes a circle of radius $R \gg a$. Find the maximum speed at which it can travel without overturning.



Solution to Problem 3.13.1

Solution

The frictional force represented in the figure is the resultant of the frictional forces where each wheel has contact with the road, which points radially inward due to the fact that the speed is constant.

The translational motion of CM is determined by:

$$\vec{F} = m\vec{a} \Rightarrow N = mg \; ; F_f = ma_N = m\frac{v^2}{R}$$

The movement is at a uniform speed. As long as the car does not overturn, the angular momentum will be constant. In other words, the car does rotate, but uniformly. Thus, if $\vec{L}_{(CM)}$ is constant,

$$\Rightarrow \frac{d\vec{L}_{(CM)}}{dt} = 0$$

It is sufficient to calculate the momentum of the forces with respect to the CM and make it equal to zero $\vec{M}_{(CM)} = 0$. Looking at the figure, we can formulate

$$\vec{M}_{(CM)} = 0 = (Na - F_f h, 0, 0) \Rightarrow Na - F_f h = 0$$

and, combining this with the translation equations, we get $v = \sqrt{gRa/h}$ ■

Problem 3.13.2. What are the conditions that a frictionless pulley must meet in order for the tension of a rope, with negligible mass, to be equal on both sides?

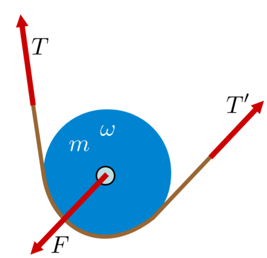


Figure for Problem 3.13.2

Solution

The moments are calculated with respect to the CM, located at the centre of the pulley:

$$\vec{L}_{(CM)} = \sum_i \vec{r}_{i(CM)} \times m_i \vec{v}_i$$

We decompose the velocity into rotational and translational velocities: $\vec{v}_i = \vec{v}_{CM} + \vec{v}_{i(CM)}$

The translation term is null: $\underbrace{\left(\sum_i m_i \vec{r}_{i(CM)}\right)}_{=0} \times \vec{v}_{CM} = 0$

Thus, we have

$$\vec{L}_{(CM)} = \sum_i \vec{r}_{i(CM)} \times m_i \vec{v}_{i(CM)} \Rightarrow \frac{d\vec{L}_{(CM)}}{dt} = \sum_i \vec{r}_{i(CM)} \times m_i \vec{a}_{i(CM)}$$

With regard to the moments of the forces with respect to the CM:

$$\vec{M}_{(CM)} = (T'R - TR)\,\hat{k}$$

Equal tensions $\Leftrightarrow \vec{M}_{(CM)} = 0 \Leftrightarrow \sum_i \vec{r}_{i(CM)} \times m_i \vec{a}_{i(CM)} = 0$

This cancellation can be due to:

1) A pulley of negligible mass ($m_i \to 0$). It can rotate and move arbitrarily.

2) A pulley in uniform rotation (null tangent accelerations): $\vec{r}_{i(CM)} \times \vec{a}_{i(CM)} = \vec{r}_{i(CM)} \times \vec{a}_{N\,i(CM)} = 0$ since $\vec{r}_{i(CM)} || \vec{a}_{N\,i(CM)}$. It can have arbitrary dimensions and mass, and it can move arbitrarily.

If the pulley has a negligible radius, care must be taken: the rotation can become very great, so much that the product does not needs to be zero! ■

3.14 Torque

Torque is a system of forces applied to a rigid body and equivalent to two equal forces, in opposite directions and with parallel lines of action; hence, of zero net force and non-zero net moment (see Figure 3.40). We will call *torque* the resultant moment of the forces.

➔ **The torque is independent of the point in space used to calculate it.**

Proof. We can see this by calculating the net moment: considering Figure 3.40,

$$\vec{M} = \vec{r}_+ \times \vec{F} - \vec{r}_- \times \vec{F} = (\vec{r}_+ - \vec{r}_-) \times \vec{F} = \vec{r}_\mp \times \vec{F}$$ ■

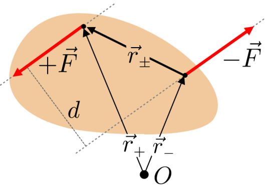


Fig. 3.40: Torque

➔ The modulus of the torque can be expressed as

$$M = Fd \tag{3.97}$$

where d is the distance between the lines of action of the torque forces. This distance is called **the torque arm**. Therefore, the torque depends on the distance between the forces (arm) and the value of the forces.

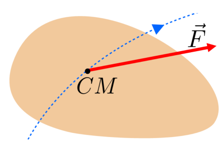


Fig. 3.41: A force $\vec{F}$ applied to the CM is the cause of a pure translation: If the body is initially at rest, the effect of the force $\vec{F}$ is to move the CM without rotating the body

It is very common to know the torques, $\vec{M}_i$, applied to a body and not the forces that constitute them. This does not prevent us from determining the motion of the body using equations (3.91) and (3.92). The torques $\vec{M}_i$ are part of the additive terms in the determination of the total moment $\vec{M}_{(C)}$. A torque normal to the plane of the paper can be represented by the symbols $\curvearrowleft$ or $\curvearrowright$, which determine the sign of the torque.

➔ **Pure translation**. In Figure 3.41 only one force is applied to the CM of a body. According to the equations of motion (3.91) and (3.92) with $C = CM$, the effect on the body, if it is initially at rest, is a translation of the CM without any rotation, which we call *pure translation*.

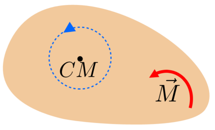


Fig. 3.42: A torque $\vec{M}$ is the cause of a pure rotation: If the body is initially at rest, the effect of the torque $\vec{M}$ is to make it rotate around CM. The CM remains at rest

➔ **Pure rotation**. In Figure 3.42 only one torque $\vec{M}$ is applied to the same body. Taking into account the equations of motion, the effect on the body, if it is initially at rest, is a rotation around the CM without it moving, a motion we call *pure rotation*.

Work, power and energy of a torque

The work done by a torque $\vec{M}$ applied to a rigid body is the work done by the constituent forces of the torque. If we express the torque with two forces $\vec{F}_+ = \vec{F}$ and $\vec{F}_- = -\vec{F}$ applied to the points in the body $\vec{r}_+$ and $\vec{r}_-$, we have $\vec{M} = (\vec{r}_+ - \vec{r}_-) \times \vec{F}$. The work will be

$$W = \int \left(\vec{F}_+ \cdot d\vec{r}_+ + \vec{F}_- \cdot d\vec{r}_- \right) = \int \vec{F} \cdot (d\vec{r}_+ - d\vec{r}_-)$$

where we have omitted the specification of the limits of integration.

The torque forces are applied to points on the body and these points can change. Now, as we have already seen in Problem 3.10.1, the displacements $d\vec{r}_\pm$ are displacements of the particles in the body, in this case in a rigid body. Considering (3.83), we can write $d\vec{r}_+ - d\vec{r}_- = d\vec{\varphi} \times (\vec{r}_+ - \vec{r}_-)$ and $\vec{F} \cdot (d\vec{r}_+ - d\vec{r}_-) = \vec{F} \cdot (d\vec{\varphi} \times (\vec{r}_+ - \vec{r}_-)) = \left((\vec{r}_+ - \vec{r}_-) \times \vec{F} \right) \cdot d\vec{\varphi}$. We obtain the expression for work done by and the power of a torque when applied to a rigid body:

$$W = \int \vec{M} \cdot d\vec{\varphi} \;\; ; \;\; P = \vec{M} \cdot \vec{\omega} \tag{3.98}$$

where $\vec{\omega} = \frac{d\vec{\varphi}}{dt}$ is the angular velocity of the body.

If the torque has the constant direction $\hat{u}$, $\vec{M} = M\hat{u}$, and the body moves along the same axis, $d\vec{\varphi} = d\varphi\, \hat{u}$, the expression for the work W and the power P are

$$W = \int_{\varphi_{\text{ini}}}^{\varphi_{\text{fi}}} M \, d\varphi \; ; \; P = M\omega \tag{3.99}$$

If the torque M depends only on φ, we can write the potential energy associated with the torque as follows

$$U = -\int M(\varphi)\, d\varphi \tag{3.100}$$

➔ If the torque is constant, we have

$$U = -M\varphi \tag{3.101}$$

➔ If the torque has the form $M = -\kappa\varphi$, as in the case of the spring shown in Figure 3.43, where κ is the spring's recovery constant (units $\mathrm{N\,m\,rad^{-1}}$), the resulting corresponding potential energy is

$$U = \frac{1}{2}\kappa\varphi^2 \tag{3.102}$$

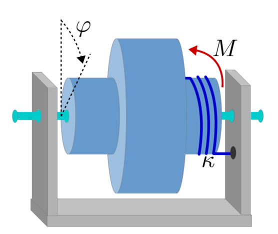
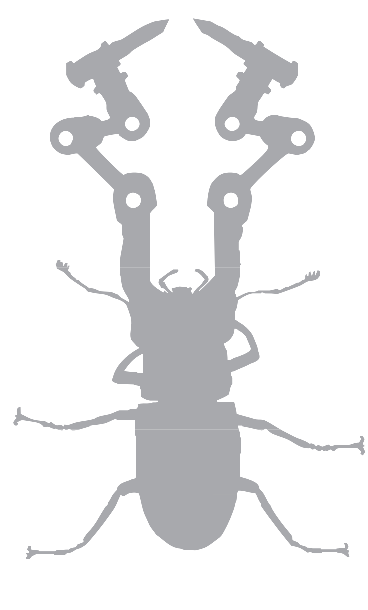


Fig. 3.43: The spring of recovery constant κ gives the body a torque of $M = -\kappa\varphi$ when it is deformed by an angle φ



→4

4 The statics of rigid bodies

Introduction

In this chapter, we will deal with the statics of rigid body and see what conditions must be met for a rigid body to be at rest. Using equivalence of force systems, we will learn how to represent the most common force systems acting on rigid bodies (see Figure 4.1) and by nulling the net moment, we will find the point at which the net force must be applied.

We will apply the motion equations by inverting what we usually take as data and unknowns. The data will be, apart from some forces, the motion of the body that we want to be at rest. We will also see how the general equation of dynamics becomes the general equation of statics, also known as the principle of virtual works.
If the constraints are ideal and the forces are conservative, we will see that the principle of virtual work can reduce the problem of equilibrium and its stability to simply studying the geometry of the potential function.

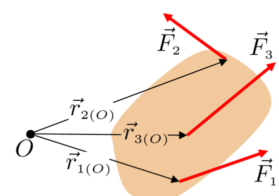


Fig. 4.1: Rigid body subjected to forces applied at different points

4.1 The statics of a body: Equilibrium conditions

A body is said to be in equilibrium when the accelerations of translation and rotation are zero. Thus, a body at rest is in equilibrium, but so is a body that moves with uniform velocity and/or has a uniform rotation. What is important is that if the initial conditions of the body are at rest and the body is at equilibrium, it will be at rest. In the previous chapters we have posed the dynamics problem by trying to find out what effects or motions caused the known forces and, therefore, the motion was unknown to us. What we want to know now is what forces cause a known effect: zero acceleration. The forces are now unknown to us, although not all of them, of course. In general, we will try to find the unknown forces that, together with the known forces and the geometric conditions (or constraints), cause the body to be in equilibrium.

We will begin with the motion equations of a rigid body (3.91, 3.92), which we have already seen in Chapter 3. If the body is in equilibrium, we have

$$\vec{F} = \sum_{i=1}^{N} \vec{F}_i = 0 \;\; ; \;\; \vec{M}_{(O)} = \sum_{i=1}^{N} \vec{r}_{i(O)} \times \vec{F}_i = 0 \tag{4.1}$$

Thus, we can state:

➔ ***Equilibrium condition of a rigid body***. *The necessary and sufficient condition for a rigid body to be in equilibrium is that the net force and torque acting on the body are zero.*

Properties and relationships

In the following we will see a series of properties and relations that will be useful to us.

➔ **a) The point O where we calculate the moments in equilibrium equation (4.1) can be any fixed point in space.**

This is a consequence of the motion equations of a body, derived in Section 3.11. There, we commented that point C, if it existed, could be a fixed point point attached to the body or, in general, the centre of mass. In the case of statics, the body is at rest and, therefore, C can be any point in a fixed reference frame.

➔ **b) Forces on a body are sliding vectors.**

This property is fulfilled whether the body is in equilibrium or not, although in the latter case it must be specified that it is fulfilled in each instant. We can make the force *slide* along its line of action by moving the point of application to any other point on this line, since the moment with respect to a fixed point O will be the same (see Figure 4.2).

Proof 1 The proof can be made by beginning with the definition of the moment of a force

$$\vec{r}_{1(O)} \times \vec{F} = (\vec{r}_{1(O)} + \vec{r}_{2(O)} - \vec{r}_{2(O)}) \times \vec{F} = \vec{r}_{2(O)} \times \vec{F} + (\vec{r}_{1(O)} - \vec{r}_{2(O)}) \times \vec{F} = \vec{r}_{2(O)} \times \vec{F}$$

■

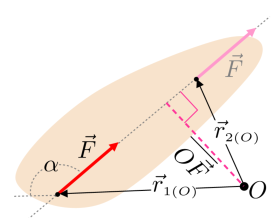


Fig. 4.2: The forces on a rigid body are sliding vectors

Proof 2 As we have seen in Chapter 2, we can calculate the moment using the corkscrew rule and (2.11). In our case, it would be

$$M_{(O)} = F\,\overline{OF}$$

Since the force $\vec{F}$, the points of application between its line of action and point O are in the same plane, the direction of the momentum will be the same and the modulus will depend only on $\overline{O\vec{F}}$, that is, on the distance between point O and the line of action of $\vec{F}$, which is independent of the point of application of $\vec{F}$ on this line (see Figure 4.2). ■

c) If only two forces act on a body in equilibrium, they must necessarily be of equal modulus, in opposite direction and have the same line of action (see Figures 4.3 and 4.4).

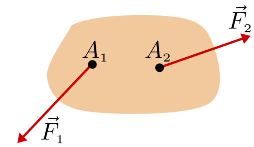


Fig. 4.3: $\vec{F}_1$ and $\vec{F}_2$ forces not known

Proof. We want the body in Figure 4.3 to be in equilibrium. The net force must be zero, $\vec{F}_2 = -\vec{F}_1$. The system in Figure 4.3 is therefore a torque. The net moment must also be zero. According to the expression for the torque (3.97), the distance between the lines of action of the forces, d, must be zero. That is, the two forces have the same coincident line of action: The force system in Figure 4.3 must be the same as the one in Figure 4.4. ■

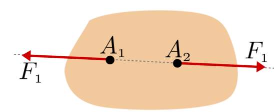


Fig. 4.4: The equilibrium conditions impose this configuration for forces $\vec{F}_1$ and $\vec{F}_2$

Problem 4.1.1. Calculate the reaction at point C (force and torque) required to keep the sphere, of negligible weight, in equilibrium.

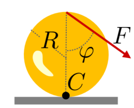


Figure for Problem 4.1.1

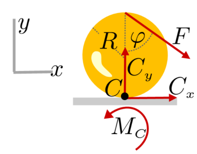
Solution

Let $\vec{C}$ be the reaction force at point C on the sphere and M_C the reaction torque. With the axes in the figure the equilibrium equations for the forces are

$$F \sin\varphi + C_x = 0$$
$$-F \cos\varphi + C_y = 0$$

and the equilibrium equation for the moments with respect to point C (remembering that we can do this for any point)

$$M_C - F \sin\varphi \, 2R = 0$$



Solution to Problem 4.1.1

We find

$$\vec{C} = -F \sin\varphi \, \hat{\imath} + F \cos\varphi \hat{\jmath} \qquad \vec{M}_C = 2RF \sin\varphi \, \hat{k}$$ ■

Problem 4.1.2. The figure shows the forces applied to a rigid body, of negligible weight, which is found to be in equilibrium. What is the value of d?

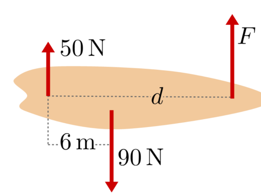


Figure for Problem 4.1.2

Solution

We do not have to calculate the force F. Let us express the equilibrium condition of the moments by taking moments with respect to the point of application of the force F

$$90\,(d - 6) - 50\,d = 0$$

and isolate d

$$d = 13.5\,\text{m}$$ ■

4.2 Weigth and centre of gravity

The weight of a body is the force exerted by the Earth's gravitational field on each of its particles (see Figure 4.5). The Earth's gravity $\vec{g}$ is considered constant due to the relatively small dimensions of the body to be treated with respect to the Earth. Unless stated otherwise, we will use the standard value $g = 9.81\,\mathrm{m/s^2}$

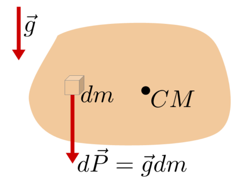


Fig. 4.5: The gravitational field acts on every particle in a body

➔ *The moment of the weight forces on a body with respect to the CM is zero.*

Proof. The position of CM relative to itself is obviously zero, $\vec{r}_{CM(CM)} = 0$. Therefore, $\frac{1}{m}\int \vec{r}_{(CM)}dm = 0$. Thus, we have

$$\vec{M}_{(CM)} = \int \vec{r}_{(CM)} \times d\vec{P} = \int \vec{r}_{(CM)} \times \vec{g}\, dm = \left(\int \vec{r}_{(CM)}\, dm\right) \times \vec{g} = 0$$

where we have used $\vec{g}$ as a constant by taking it out of the integral. ■

This important property of the CM means that in this context it is referred to as the ***centre of gravity***, CG, of the body.

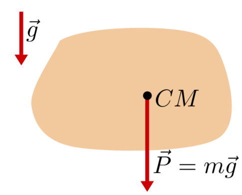


Fig. 4.6: A very simple system of equivalent forces is that of the weight applied to the $CG = CM$

For mechanical purposes, and in accordance with what we know about equivalent force systems explained at the end of Section 3.11, we can substitute the forces of each of the body parts by the net weight force applied to the CM or CG (see Figure 4.6). This weight force will have the same net force and the same (zero) net torque with respect to the CM as the weight force system $d\vec{P} = \vec{g}dm$, of each differential mass, dm, of the body. We must remember that this substitution can be made because the body is a rigid body.

4.3 Forces on bodies due to gravitating fluids: Archimedes principle

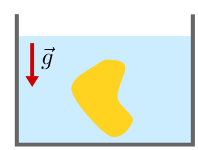


Fig. 4.7: Body immersed in a fluid

In this section, we will study the forces that fluids exert on bodies and their application points.

Consider a body immersed in a fluid (see Figure 4.7). As with any contact, there will be normal forces and tangential forces. In a general situation, static or not, if the fluid is non-viscous, the different fluid layers slide; there is no friction either between the fluid layers or between the fluid and the body. Looking at Figure 4.8, although the surface of the body is not smooth, the non-viscosity of the fluid means that there is no frictional force between the body and the fluid. Therefore, tangential forces at the contact surface are zero and we take into account only the normal forces that act in a distributed manner on the contact surface (see Figures 4.8 and 4.9). If the situation we are dealing with is static and even though the fluid is viscous, the viscous frictional forces (proportional to the velocity) will be zero and the force will be normal on the surface.

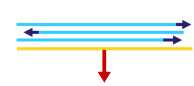


Fig. 4.8: Even if the surface of the body is rough, if the fluid is non-viscous it will not cause tangential forces

Pressure concept

We define a fluid's **pressure** p as the normal force per unit area acting on the surface of a body immersed in the fluid (see Figure 4.9)

$$p = \frac{dF}{dS} \tag{4.2}$$

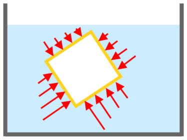


Fig. 4.9: Distribution of normal forces on a body caused by a gravitating fluid

If we consider smaller and smaller cubes, the concept of pressure is independent of the existence of a material surface. It will depend on the characteristics of the fluid and the point where we want to calculate it (see Figure 4.10).

In Figure 4.10, the pressure at the cube's location is p, even if the cube is not there.

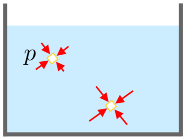


Fig. 4.10: Pressure exists even if there is no submerged body

Units of pressure

In the International System of Units, pressure is measured in pascals (Pa)

- $1\,\text{Pa} = 1\,\text{N/m}^2$

Other widely used units are:

- milibar (mbar): $1\,\text{mbar} = 1\,\text{hPa}$
- atmosphere (atm): $1\,\text{atm} = 1.013 \times 10^5\,\text{Pa}$
- millimetres of mercury (mmHg): $750\,\text{mmHg} = 1\,\text{atm}$

Fluid pressure in a uniform gravitational field

Consider a differential volume of fluid in the form of a vertical cylinder of base S and height dz (see Figure 4.11). z is the **depth**, a coordinate that has the direction of $\vec{g}$ with origin at the fluid level. The force that this portion of fluid exerts on the base of the cylinder is equal to the weight of the fluid inside the cylinder.

$$dF = dm\, g = \rho S dz\, g$$

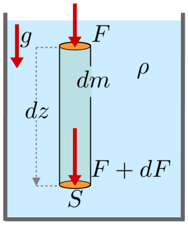


Fig. 4.11: Differential fluid volume in the form of a vertical cylinder

Therefore,

$$dp = \rho g\, dz \tag{4.3}$$

If the fluid is liquid, we can consider it as incompressible, i.e., its density ρ will be constant at all points within the liquid. Integrating the above equation, we obtain the difference of pressures between two points at different heights or depths:

$$\Delta p = \rho g\, \Delta z \tag{4.4}$$

In a homogeneous fluid ($\rho = \text{ct}$), whether closed or not, points of equal height have the same pressure. The pressures at equal height may be different when more than one fluid is in contact within the container or, in general, when the fluid is not homogeneous.

Since the density of a gas is much smaller than that of a fluid, the pressure difference between two points at different heights in gases will be much smaller than in liquids. Thus, for example, if we have a container where liquids and gases coexist, the pressure differences in the case of gases will be negligible.

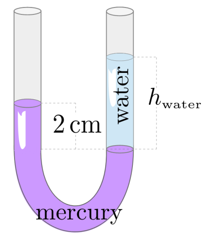
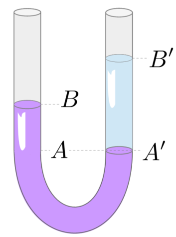


Figure for Problem 4.3.1

Problem 4.3.1. We fill a section of a **U**-shaped tube, open at both ends, with water and mercury. If the difference between the mercury levels is $h_{\text{Hg}} = 2\,\text{cm}$, calculate the height of the column of water h_{water}.

Dades: $\rho_{\text{Hg}} = 13 \times 10^3\,\text{kg}\,\text{m}^{-3}$, $\rho_{\text{water}} = 10^3\,\text{kg}\,\text{m}^{-3}$



Solution to Problem 4.3.1

Solution
Since the pressure differences in air are negligible compared to those in liquids, the two levels B and B' in contact with air have the same pressure. The two points A and A' connected to the mercury have the same pressure. Thus, the pressure caused by the height h_{Hg} of mercury will be equal to that caused by the height h_{water} of water:

$$\rho_{\text{Hg}} g h_{\text{Hg}} = \rho_{\text{water}} g h_{\text{water}}$$

isolating h_{water}

$$h_{\text{water}} = \frac{\rho_{\text{Hg}} h_{\text{Hg}}}{\rho_{\text{water}}} = 26\,\text{cm} \qquad \blacksquare$$

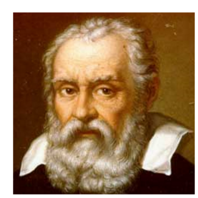


Fig. 4.12: Archimedes of Syracuse (287 BC to 212 BC)

Archimedes principle

➔ ***Archimedes principle.*** *Any body immersed in a gravitating fluid experiences a vertical upward **buoyant force**, E, equal to the weight of the fluid it displaces:*

$$E = P_{\text{fluid}} \tag{4.5}$$

*The buoyancy E is applied at a point, the **centre of buoyancy** CE, located at the centre of mass of the displaced fluid, CM_{fluid}.*

$$CE = CM_{\text{fluid}} \tag{4.6}$$

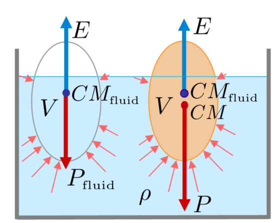


Fig. 4.13: On the right, the distribution of forces caused by the fluid on a body immersed in it. On the left, the same distribution that would act on a surface having the same shape as the body on the right

Proof. Figure 4.13 shows, on the right, a half-submerged body occupying a volume V of fluid and, on the left, a surface with the same shape as the original body, which encloses a part of the fluid, V, equal to that displaced by the original body. Since this part of the fluid V is in equilibrium, the weight of the enclosed fluid, P_{fluid}, will be equal to the net force of the distributed forces due to the pressure, i.e., the buoyancy E: $E = P_{\text{fluid}}$. If, in addition, the fluid is non-compressible, we will have $E = \rho g V$.

The point of application of E will be the centre of mass of the fluid enclosed in the surface (CM_{fluid}). Since E applied at CE is exclusively a consequence of the forces that the fluid exerts on the surface, it will be the same as what the original body receives with the same shape as the surface and equally submerged, although its weight will be different. ■

Problem 4.3.2. In a container filled with water of density $\rho_a = 1\,\text{g/cm}^3$ and oil of density $\rho_{\text{oil}} = 0.8\,\text{g/cm}^3$, a cylinder of height $H = 15\,\text{cm}$ and density $\rho = 0.9\,\text{g/cm}^3$ is floating in equilibrium with its axis in a vertical position. If we see that the cylinder protrudes at a height of $z = 0.8\,\text{cm}$ above the upper level of the oil, calculate the heights x and y of the submerged cylinder in, respectively, the water and the oil.

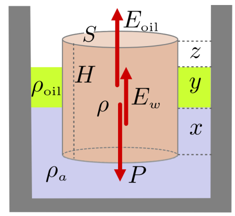


Figure for Problem 4.3.2

Solution

The vertical force balance equation is $E_{\text{oil}} + E_w - P = 0$, where $E_{\text{oil}} = \rho_{\text{oil}} g\ Sy$ is the buoyancy due to the oil, $E_w = \rho_a g\ Sx$ is the buoyancy due to the water and $P = \rho g\, SH$is the weight of the cylinder. $H = x + y + z$ must also be fulfilled. Substituting, we obtain the system:

$$\begin{cases} \rho_{\text{oil}} y + \rho_a x - \rho H = 0 \\ H = x + y + z \end{cases}$$

which, taking into account the data of the statement, we can solve in x and y. We obtain:

$x = 10.7\,\text{cm}$ and $y = 3.5\,\text{cm}$ ■

Forces on flat surfaces

We want to calculate the net force and the point of application of the forces of an incompressible fluid on a flat surface with a certain angle of inclination with respect to the horizontal plane.

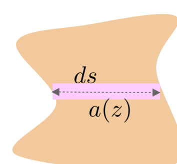


Fig. 4.14: Surface element ds, with a horizontal width $a(z)$ of a surface immersed in a fluid

Looking at Figures 4.14 and 4.15, the force dF that the liquid exerts on the surface ds is $dF = pdS$, with $p = \rho g z$. If we take into account that ρ and g are constants, we can write

$$F = \rho g \int z\, ds \tag{4.7}$$

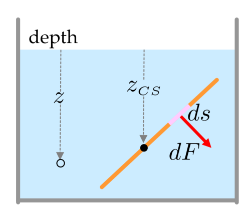


Fig. 4.15: Force, dF, that the fluid makes on the surface element ds and depth of the CS, z_{CS}

The depth z (see Figure 4.15) of the centre of symmetry CS, z_{CS} of the surface is

$$z_{CS} = \frac{1}{S} \int z ds \tag{4.8}$$

Thus, we obtain, by substituting the integral of (4.7) by using (4.8)

$$F = \rho g z_{CS} S \tag{4.9}$$

Therefore, we can calculate the net force on a flat surface by multiplying the pressure at the z_{CS} depth by the area of the flat surface.

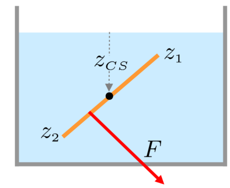


Fig. 4.16: Net force, F, of the fluid on the surface of constant horizontal width and depth z_1 and z_2 at its extremities

If the width of the surface, a, is constant, $S = aL$, it is simple to calculate the depth z_{CS} of its centre of symmetry (see Figure 4.16):

$$z_{CS} = z_1 + \frac{1}{2}(z_2 - z_1) = \frac{1}{2}(z_2 + z_1)$$

Substituting in (4.9), we get

$$F = \rho g \, \frac{1}{2}(z_2 + z_1)\, aL \tag{4.10}$$

The point of application of the net force x_F (see Figure 4.17) is found by imposing (if possible) that the net moment with respect to this point is zero. Looking at Figure 4.17, the condition of null momentum with respect to x_F can be expressed as

$$\int (x - x_F)\, dF = 0 \tag{4.11}$$

from which we isolate x_F

$$x_F = \frac{1}{F} \int x \, dF \tag{4.12}$$

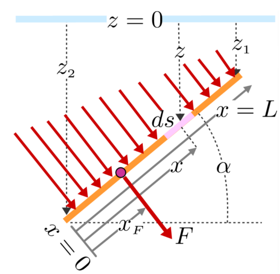


Fig. 4.17: Distribution of the forces dF on each element ds as a function of the depth and coordinate x that locates each ds on the surface

If we take into account that we are studying the case $a =$ ct, $ds = adx$ and $dF = \rho g z \, dx$ while also taking into account (4.10), (4.12) can be written

$$x_F = \frac{2}{L(z_2 + z_1)} \int x \; z \; dx \tag{4.13}$$

In the above integral, the two variables x and z are mixed and not independent. The relationship between these variables will be linear, $z = Ax + B$, and we can find the constants A and B by means of the conditions $x = 0 \Rightarrow z = z_2$ and $x = L \Rightarrow z = z_1$. Alternatively, looking at Figure 4.17, we can express the sine of the α angle as $\sin\alpha = \frac{z_2 - z_1}{L} = \frac{z_2 - z}{x}$. We obtain

$$z = z_2 - \frac{z_2 - z_1}{L} x \tag{4.14}$$

and substituting in (4.13) with the limits of integration for the variable x from $x = 0$ to $x = L$

$$x_F = \frac{2}{L(z_2 + z_1)} \int_0^L x \; (z_2 - \frac{z_2 - z_1}{L} x) \; dx \tag{4.15}$$

by performing the integral, we finally obtain

$$x_F = \frac{L(z_2 + 2z_1)}{3(z_2 + z_1)} \tag{4.16}$$

Problem 4.3.3. A 1 m wide vessel has a gate $\overline{AB}$ of negligible mass that is supported by forces F_1 and F_2, which are perpendicular to the gate, as shown in the figure. Find the minimum values of these forces that cause the gate to be in equilibrium.

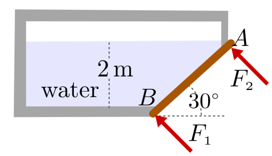


Figure for Problem 4.3.3

Solution

The width is $a = 1$ m. We can find $\overline{AB}$:

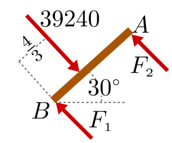
$$\overline{AB} = L = \frac{2}{\sin 30^\circ} = 4 \text{ m}$$

Applying the expression (4.9), with $z_1 = 0$ and $z_2 = 2$ m, we get

$$F_{\text{water}} = 10^3 \cdot 9.81 \cdot \frac{1}{2} \cdot 2 \cdot 1 \cdot 4 = 39240 \text{ N}$$

The point of application is found by means of (4.16)

$$x_F = \frac{4 \cdot (2 + 2 \cdot 0) \cdot 1}{3(2+0)} = \frac{4}{3} \text{ m}$$



Solution to Problem 4.3.3

Now we apply the force and moment equilibrium equations. Note that we want the minimum values and therefore the reactions of the supports at points A and B on the gate will be zero. The moments are calculated with respect to B:

$$\begin{cases} \sum \vec{F}_i = 0 \ : \quad F_1 + F_2 - F_{\text{water}} = 0 \\ \sum \vec{M}_{(B)i} = 0 \ : F_1\, L - \ F_{\text{water}}\, x_F = 0 \end{cases}$$

We solve for F_1 and F_2

$$F_1 = 13080 \text{ N} \ ; \ F_2 = 26160 \text{ N}$$

■

4.4 Constraints and reaction forces

If a body in equilibrium has supports or, in general, constraints (ropes, contacts, rollers, joints, etc.) and we want to solve the equilibrium equations, what we do is replace the constraints with reaction forces. A diagram in which the body is presented and where all the constraints have been replaced by the corresponding reaction forces is called a ***free body diagram*** (FBD). The table on page 338 provides a list of possible constraints for a body with the corresponding reaction forces.

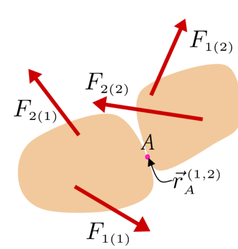


Fig. 4.18: External forces $F_{i(1)}$ and $F_{i(2)}$ on bodies 1 and 2, respectively, representing a contact at A

4.5 Statics of N rigid bodies

If we have N bodies, $i = 1, 2, 3 \ldots N$, each of them subjected to external forces and with possible contacts between them, we can make the free body diagram for the whole or for each body separately. In the case that we want the free body diagram to represent a set of bodies, the contact forces would be internal and would cancel each other out two by two and, thus, we would not have to include them in the diagram or in the equilibrium equations (see Figure 4.18).

If we want to represent each body separately, the contact forces will be forces external to the body on which they act. Contact forces obey the law of action and reaction $\vec{F}_{A(ji)} = -\vec{F}_{A(ij)}$, and will have opposite direction if we apply them to one body or the other (see Figure 4.19).

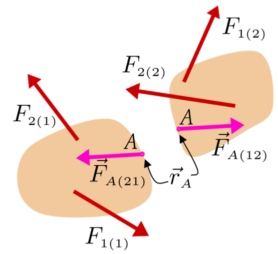


Fig. 4.19: If we do the FBD of each body, we have to include the reaction forces at contact A, $\vec{F}_{A(ji)} = -\vec{F}_{A(ij)}$

Each of the N bodies will satisfy the equilibrium equations, $\vec{F}_i = 0$ and $\vec{M}_{i(P_i)} = 0$, which must include the reaction forces.

Suppose, in the presence of gravity, we have two blocks of masses m_A and m_B, one on top of the other and on a plane inclined at an angle θ (see Figure 4.20). When we represent the forces in the diagram of the set of the two blocks without the inclined plane, we draw the external forces, such as the weights of blocks A and B, the contact forces between the plane and block B, decomposed into normal forces and tangential or frictional forces.

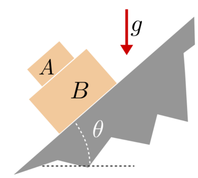


Fig. 4.20: Two blocks, A and B, in contact on an inclined plane

When representing the diagram of each block separately we must also include the contact forces between A and B. As can be seen in Figure 4.21, both the normal and friction forces between blocks A and B point in opposite directions when represented in each block.

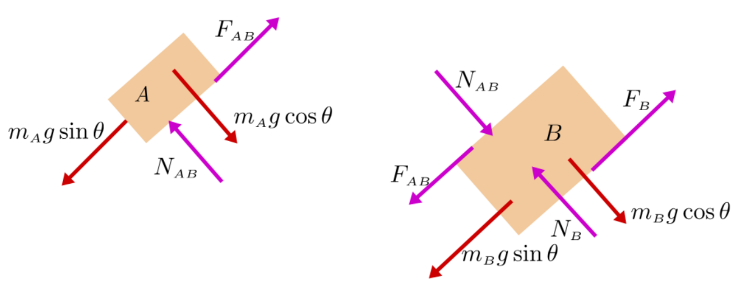


Fig. 4.21: Free body diagrams of blocks A and B

Problem 4.5.1. The bar in the figure is 80 cm long, has a mass of 20 kg, and is joined without friction at fixed point A. What torque (moment) must the crank handle make on the bar to support the 100 kg block and what force acts on the bar at point A (two components)?

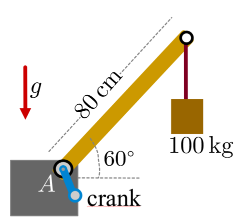
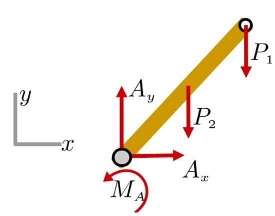


Figure for Problem 4.5.1

Solution

The 100 kg block produces a tension in the rope of value $P_1 = 100 \cdot 9.81 = 980\,\text{N}$ and the weight of the bar $P_2 = 20 \cdot 9.81 = 196.2\,\text{N}$, which is applied at the CM of the bar. We use the equilibrium equations for the bar and take the moments with respect to point A (see Figure):

$x)\ A_x = 0$



Solution to Problem 4.5.1

$y)\ A_y - P_1 - P_2 = 0$

$A)\ M_A - P_1 0.8 \cos 60° - P_2 0.4 \cos 60° = 0$

from which we get

$$A_x = 0$$
$$A_y = 1177.2\,\text{N}$$
$$M_A = 431.64\,\text{N m}$$

■

Problem 4.5.2. The two homogeneous bars have different lengths and masses. The joints A and B and the pulley D are frictionless. $\overline{CD} \gg \ell_1 + \ell_2$ and therefore the rope $\overline{CD}$ is always horizontal. Find the angles α and β in the equilibrium system configuration using the moment equilibrium equations.

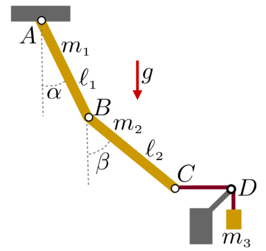


Figure for Problem 4.5.2

Solution

Taking into account the body hanging from the rope, the tension of the rope is $T = m_3 g$.

We must be careful to state the force balance equations for the system formed by the two rods, considering that we are not interested in finding the reactions of joints A and B. We will use the moment equilibrium equations in an appropriate way. If we consider only the bar $\overline{BC}$, the moment equilibrium equation with respect to B is

$$m_3 g \ell_2 \cos\beta - m_2 g \frac{\ell_2}{2} \sin\beta = 0 \qquad (1)$$

from where we obtain $\tan\beta = \dfrac{2m_3}{m_2}$

If we now consider the two bars, the moment equilibrium equation with respect to A is

$$m_3 g(\ell_2 \cos\beta + \ell_1 \cos\alpha) - m_2 g\left(\ell_1 \sin\alpha + \frac{\ell_2}{2}\sin\beta\right) - m_1 g \frac{\ell_1}{2} \sin\alpha = 0$$

which, taking into account (1), gives

$$m_3 \cos\alpha - m_2 \sin\alpha - m_1 \frac{1}{2} \sin\alpha = 0$$

and, isolating $\tan\alpha$, we find $\tan\alpha = \dfrac{m_3}{\frac{m_1}{2} + m_2}$ ■

4.6 Virtual work principle

Consider a system of bodies with ideal constraints and subjected to directly applied forces, $\vec{F}_a$, which may include, for example, gravity or spring force, as in Figure 4.22. Let us assume that we know the directly applied forces $\vec{F}_a$, meaning that we know these forces as a function of the point where they are applied. If $\vec{F}_1$ and $\vec{F}_2$ are the weight or a constant force, then knowing these forces is to know the numerical

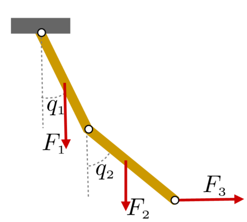


Fig. 4.22: Two joined bars subjected to the directly applied forces $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$

value of the modulus (as well as the direction). In the case of $\vec{F}_3$, we know it as a function of the point of application $\vec{r}_3$. What is unknown in this approach is the system configuration. In the case of the figure (this is a system with two degrees of freedom), the unknowns are the angles q_1 and q_2 formed by the two bars with the vertical. The solution can be found by applying the static equations to the two bodies which would reveal all the reaction forces as new unwanted unknowns. Remembering that ideal reaction forces do not appear in the general equation of dynamics, we can use this in our approach to the case of statics.

➔ ***General equation of statics***. *The necessary and sufficient condition for a system of bodies with ideal constraints to be in equilibrium in a given position is that, for any virtual displacement from this position, the sum of the virtual work of the directly applied forces (those which are not reactions to the constraints) is zero:*

$$\sum_{a=1} \vec{F}_a \cdot \delta\vec{r}_a = 0 \tag{4.17}$$

Proof. Starting from the general equation of dynamics (3.64) and taking into account that at equilibrium $\vec{a}_i = 0$, (4.17) is verified.

Note that instead of using the index $i = 1...N$, we use the index a. This is because suppressing the accelerations eliminates the need to account for what happens in all the m_i. Since the forces will generally be applied to only a few particles, we can use a, which labels only these forces and the particles on which they act, $\vec{r}_a$. ■

Expression (4.17) is commonly referred to as the **virtual work principle** (VWP). In this equation, external forces are included, e.g., the weights applied to the CM of each body, but not the ideal reaction forces, as discussed in Section 3.9.

If we know the constraints, we can know the possible motions and therefore the degrees of freedom, L, of the system. The number of degrees of freedom will be equal to the number of parameters or independent variables. Thus, looking at Problem 4.6.2 on page 129 as an example, because it is a set of two rods joined at a fixed end, the system has two degrees of freedom and we can therefore give its position with the values anglesα and β, which are the parameters or variables that must be taken into account in the equation.

If the system has L independent parameters q_i, with $i = 1...L$, and we know the constraints in the form of $\vec{r}_a = \vec{r}_a(q_1, q_2, ...)$, the expression for the virtual displacements will be

$$\delta\vec{r}_a = d\vec{r}_a = \sum_{i=1}^{L} \frac{\partial \vec{r}_a}{\partial q_i} dq_i$$

Applying this to the VWP equation (4.17), we obtain a number of L equilibrium equations that is equal to the number of unknown q_i.

$$\sum_{a=1} \vec{F}_a \cdot \frac{\partial \vec{r}_a}{\partial q_i} = 0 \tag{4.18}$$

The solutions $q_i = q_{eq\,i}$ are the equilibrium positions of the system.

Problem 4.6.1. If the system in the figure is in equilibrium, determine the relationship between the forces F and P as a function of the angle θ of equilibrium.

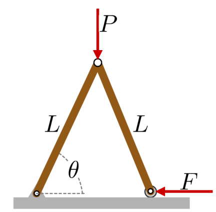


Figure for Problem 4.6.1

Solution

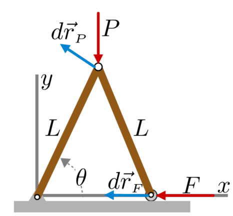
The system is conservative because the forces $\vec{P}$ and $\vec{F}$ are constant and because the constraints have ideal reactions. Thus, the VWP is expressed as

$$\vec{P} \cdot d\vec{r}_P + \vec{F} \cdot d\vec{r}_F = 0 \tag{1}$$

where $d\vec{r}_P$ and $d\vec{r}_F$ are the virtual displacements of the points of application of the forces $\vec{P} = (0, -P)$ and $\vec{F} = (-F, 0)$.

The system has a single degree of freedom expressed by the angle parameter θ. Looking at the figure and taking into account the chosen axes, we see that $\vec{r}_P = (L\cos\theta, L\sin\theta)$ and $\vec{r}_F = (2L\cos\theta, 0)$. By differentiating, we get the possible displacements

$$d\vec{r}_P = (-L\sin\theta, L\cos\theta)\ d\theta$$
$$d\vec{r}_F = (-2L\sin\theta, 0)\ d\theta$$



Solution to Problem 4.6.1

Substituting the forces and displacements in (1), we have $(-PL\cos\theta + F2L\sin\theta)d\theta = 0$ and as $d\theta$ is any possible displacement, we are given

$$-PL\cos\theta + F2L\sin\theta = 0$$

from which we have

$$\tan\theta = \frac{P}{2F}$$ ■

Problem 4.6.2. The two homogeneous bars have different lengths and masses. The joints A and B and the pulley D are frictionless. $\overline{CD} \gg \ell_1 + \ell_2$ and therefore the rope $\overline{CD}$ is always horizontal. Find the angles α and β in the equilibrium configuration of the system using the virtual work principle.

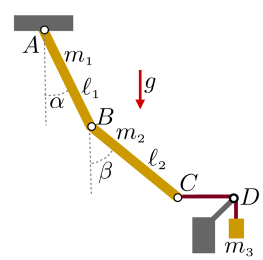


Figure for Problem 4.6.2

Solution

Taking into account the body hanging from the rope, the tension of the rope is $T = m_3 g$.

We will use the (x, y) reference frame such that the origin is at point A, the x-axis is horizontal and extends to the right, and the y-axis is vertical and extends downwards.

The VWP expression will be $\vec{F}_1 \cdot d\vec{r}_1 + \vec{F}_2 \cdot d\vec{r}_2 + \vec{T}_3 \cdot d\vec{r}_3 = 0$ which, making explicit $\vec{F}_1 = (0, m_1 g)$, $\vec{F}_2 = (0, m_2 g)$ and $\vec{T} = (m_3 g, 0)$, results in

$$m_1 dy_1 + m_2 dy_2 + m_3 dx_3 = 0 \tag{1}$$

where we have already eliminated g from the expression. The displacements $d\vec{r}_1$, $d\vec{r}_2$ and $d\vec{r}_3$ are not independent, but they depend on the two angles α and β:

$$\begin{aligned}
y_1 &= \tfrac{\ell_1}{2}\cos\alpha \;\Rightarrow\; dy_1 = -\tfrac{\ell_1}{2}\sin\alpha\, d\alpha \\
y_2 &= \ell_1\cos\alpha + \tfrac{\ell_2}{2}\cos\beta \;\Rightarrow\; dy_2 = -\ell_1\sin\alpha\, d\alpha - \tfrac{\ell_2}{2}\sin\beta\, d\beta \\
x_3 &= \ell_1\sin\alpha + \ell_2\sin\beta \;\Rightarrow\; dx_3 = \ell_1\cos\alpha\, d\alpha + \ell_2\cos\beta\, d\beta
\end{aligned}$$

Substituting in (1), we obtain

$$\begin{aligned}
-m_1\frac{\ell_1}{2}\sin\alpha\, d\alpha \quad &+ \quad m_2\left(-\ell_1\sin\alpha\, d\alpha - \frac{\ell_2}{2}\sin\beta\, d\beta\right) \\
&+ \quad m_3(\ell_1\cos\alpha\, d\alpha + \ell_2\cos\beta\, d\beta) = 0 \qquad (2)
\end{aligned}$$

If we now take into account that $d\alpha$ and $d\beta$ are independent possible displacements, we can consider (2) with $d\alpha \neq 0$ and $d\beta = 0$:

$$-m_1\frac{1}{2}\sin\alpha - m_2\sin\alpha + m_3\cos\alpha = 0 \qquad (3)$$

and also $d\alpha = 0$ and $d\beta \neq 0$

$$-m_2\frac{1}{2}\sin\beta + m_3\cos\beta = 0 \qquad (4)$$

From (3) and (4), we obtain $\tan\alpha = \dfrac{m_3}{\frac{m_1}{2} + m_2}$ and $\tan\beta = \dfrac{2m_3}{m_2}$, a result already obtained in Problem 4.5.2 using the moment equilibrium equations. ■

4.7 Equilibrium and stability in conservative systems

Equilibrium

Let us now consider that we have a conservative system, that is, with the same conditions as in the previous section and, in addition, all the forces applied directly $\vec{F}_a$ are conservative, that is, each force has an associated potential energy and, therefore,

$$\sum_{a=1} \vec{F}_a \cdot d\vec{r}_a = -dU$$

where U is the potential energy of the system. The VWP can now be written as

$$dU = 0 \qquad (4.19)$$

Making this explicit, the equilibrium equations can be written as a geometric condition in the potential function:

➔ *The equilibrium position $q_{\text{eq}} = \{q_{\text{eq}1}, q_{\text{eq}2} \dots q_{\text{eq}L}\}$ of a conservative system corresponds to the point at which the potential energy is extreme:*

$$\left.\frac{\partial U}{\partial q_i}\right|_{q=q_{\text{eq}}} = 0 \quad i = 1 \dots L \qquad (4.20)$$

Stability

If a system is in the equilibrium position q_{eq}, with $E_c = 0$, its energy is $E = U_{ext}$, where $_{ext}$ stands for extreme. If we provide kinetic energy $E_{c\ ini}$, which is always positive, the system will move with mechanical energy $E = U_{ext} + E_{c\ ini}$ =constant.

In Figures 4.23 and 4.24, we can see the graphs of the potential energy (blue) and mechanical energy (pink) around an extreme for any system. It could be a roller coaster. In this case, the shape of the guide coincides with the shape of the potential function.

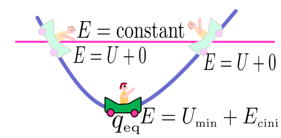


Fig. 4.23: The carriage is located at the bottom of the roller coaster. If we push it a little, i.e., if we give it a little $E_{c\ ini}$, the mechanical energy will be $E = U_{min} + E_{c\ ini}$ and it will be constant. The carriage will move around the minimum between the two positions where $E = U$, because, beyond that, it would have to be $E_c < 0$ to fulfil $E =$ constant, which cannot be the case

Depending on the type of equilibrium q_{eq} position, we can say:

- q_{eq} **is a stable equilibrium position** when $U(q_{eq})$ is a minimum. In this case we have that $U > U_{min}$ and since $E = U_{min} + E_{c\ ini}$ = constant, the system cannot move very far away from position q_{eq}, it can go until $E_c = 0$.

- q_{eq} **is an unstable equilibrium position** when $U(q_{eq})$ is either a maximum or an inflection point. If it is a maximum, we have $U < U_{max}$ and since $E = U_{max} + E_{c\ ini}$ =constant, the system moves far away from position q_{eq}, $E_c \neq 0$ and increase. If it is an inflection point and goes towards zone $U > U_{inf}$, it behaves as in the case of the minimum, reaching point $E_c = 0$ and returning to zone $U < U_{inf}$ and from here the same thing happens as in the case of the maximum: $E_c \neq 0$ and it increase.

- q_{eq} **is an indifferent equilibrium position** when in a finite environment of q_{eq}, U is constant.

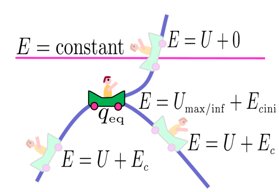


Fig. 4.24: The carriage is located at either a maximum or an inflection point on the roller coaster. If we push it a little, i.e., if we give it a little $E_{c\ ini}$, the mechanical energy will be $E = U_{max/inf} + E_{c\ ini}$. If it is a maximum, E_c will increase as it will be in a zone where $U < U_{max}$ and E =constant must be fulfilled. If it is an inflection point and it goes towards a zone where $U > U_{inf}$, it will not be able to go beyond $E = U$, that is, $E_c = 0$ and it will return downwards, now increasing its E_c

For a system with a single degree of freedom q, $q = q_{eq}$, the equilibrium condition (4.20) reduces to

$$\left.\frac{dU}{dq}\right|_{q=q_{eq}} = 0 \tag{4.21}$$

To study the stability of the position q_{eq}, we analyse the second derivative at point $q = q_{eq}$.

- If it happens that

$$\left.\frac{d^2U}{dq^2}\right|_{q=q_{eq}} > 0 \tag{4.22}$$

then $q = q_{eq}$ is a stable equilibrium point.

- If it happens that

$$\left.\frac{d^2U}{dq^2}\right|_{q=q_{eq}} < 0 \tag{4.23}$$

then $q = q_{eq}$ is an unstable equilibrium point.

➔ If it happens that

$$\left.\frac{d^2U}{dq^2}\right|_{q=q_{eq}} = 0 \tag{4.24}$$

then the sign of the higher order derivatives must be examined:

➔ **The equilibrium is stable** if the order of the first non-zero derivative is even and if its sign is positive.

➔ **The equilibrium is indifferent** if all successive derivatives are zero.

➔ **The equilibrium is unstable** in all other cases.

Problem 4.7.1. The two homogeneous bars have different lengths and masses. The joints A and B and the pulley D are frictionless. $\overline{CD} \gg \ell_1 + \ell_2$ and therefore the rope $\overline{CD}$ is always horizontal. Find the angles α and β in the equilibrium system configuration, bearing in mind that the system is conservative.

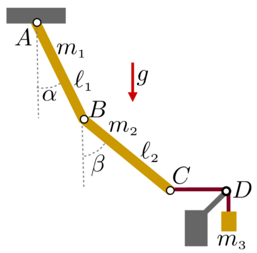


Figure for Problem 4.7.1

Solution

The potential energy of the system as a function of the two degrees of freedom, angles α and β, is

$$U = -m_1 g\frac{\ell_1}{2}\cos\alpha - m_2 g\left(\ell_1\cos\alpha + \frac{\ell_2}{2}\cos\beta\right) - m_3 g(\ell_1\sin\alpha + \ell_2\sin\beta) \quad (1)$$

The two equilibrium equations are

$$\frac{\partial U}{\partial \alpha} = 0 \Rightarrow \frac{1}{2}m_1\sin\alpha + m_2\sin\alpha - m_3\cos\alpha = 0$$
$$\frac{\partial U}{\partial \beta} = 0 \Rightarrow \frac{1}{2}m_2\sin\beta - m_3\cos\beta = 0$$

from which we obtain $\tan\alpha = \dfrac{m_3}{\frac{m_1}{2} + m_2}$ and $\tan\beta = \dfrac{2m_3}{m_2}$, a result already obtained in Problems 4.5.2 and 4.6.2 by other methods. ■

Problem 4.7.2. In the system shown in the figure, the bars have negligible mass. Neither the joints A and B nor the roller C, of negligible mass, have friction. The spring, of negligible mass, has a recovery constant k and a natural length $L_k = 0.6\,\text{m}$. From joint A hangs a block of mass $m = 75\,\text{kg}$. Find the value of k so that the equilibrium angle θ is $\theta_{eq} = 35°$. **Other data:** $L = 1.2\,\text{m}$, $L_0 = 0.85\,\text{m}$

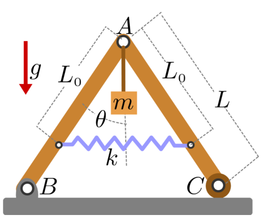


Figure for Problem 4.7.2

Solution

The potential energy of the system is

$$U(\theta) = mgL\cos\theta + \frac{1}{2}k(2L_0\sin\theta - L_k)^2$$

and the equilibrium condition is

$$\frac{dU(\theta)}{d\theta} = -mgL\sin\theta + k(2L_0\sin\theta - L_k)2L_0\cos\theta = 0$$

The spring recovery constant k must satisfy this condition for the angle $\theta_{\text{eq}} = 35°$. We obtain:

$$k = \frac{mgL\tan\theta_{\text{eq}}}{2L_0(2L_0\sin\theta_{\text{eq}} - L_k)} = 969.54\,\text{N/m}$$ ■

Problem 4.7.3. The homogeneous bar of length L and mass m can slide without friction. The spring is relaxed when $\theta = 90°$. What value must the spring recovery constant k have so that $\theta = 45°$ is an equilibrium position?

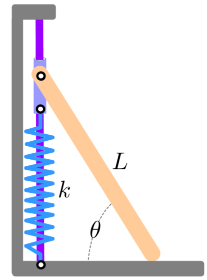


Figure for Problem 4.7.3

Solution

The potential energy of the system is

$$U(\theta) = mg\frac{L}{2}\sin\theta + \frac{1}{2}k(L - L\sin\theta)^2$$

and the equilibrium condition is

$$\frac{dU(\theta)}{d\theta} = mg\frac{L}{2}\cos\theta - kL^2(1 - \sin\theta)\cos\theta = 0$$

The spring recovery constant k must satisfy this condition for the angle $\theta_{\text{eq}} = 45°$

$$\frac{dU}{d\theta}(\theta_{\text{eq}}) = 0$$

We obtain:

$$k = \frac{mg}{2L\left(1 - \frac{1}{\sqrt{2}}\right)}$$ ■

→5

5 Dynamics of a rigid body in a plane

Introduction

A body is in plane motion if the direction of the axis of rotation remains constant. Every particle of the body moves in a plane normal to the axis of rotation. Of all these planes, the one containing the body's centre of mass is called the ***plane of motion*** (see Figure 5.1).

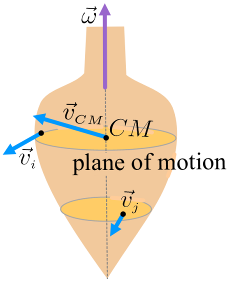


Fig. 5.1: Motion of a rigid body in a plane

According to this definition, the angular velocity $\vec{\omega}$ and angular acceleration $\vec{\alpha}$ in a body's plane motion will at all times be parallel to each other and perpendicular to the plane of motion. The motion of a rigid body is in a plane because:

a) There are external constraints that force it to move in a plane.

Example 1: A pulley rotating about a fixed axis (see Figure 5.2).

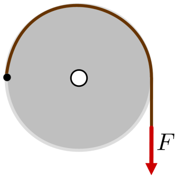


Fig. 5.2: Rigid body rotating about an axis

Example 2: A flat sheet moving in a plane (see Figure 5.3).

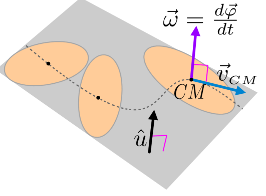


Fig. 5.3: Flat sheet moving in a plane

b) The body has plane symmetry and the external forces are on its plane of symmetry as well as the initial velocities.

Example 3: A flat sheet thrown into a vertical plane coincident with its own plane (see Figure 5.4).

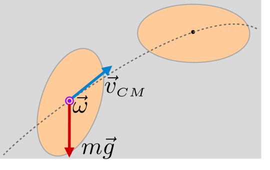


Fig. 5.4: Flat sheet moving in a vertical plane

Example 4: A homogeneous sphere moving down an inclined plane starting from rest (see Figure 5.5).

c) The forces external to the body are equivalent to a net force passing either through the CM and a zero net moment or in the direction of the axis of the body. The initial rotation of the body has the direction of this axis.

Example 5: A disc thrown with an initial rotation about the axis of the disc in the presence of gravity. Its CM performs a parabola and the direction of rotation is constant (see Figure 5.6).

Example 6: in the presence of gravity, a spinning top rotates along a vertical axis passing through the CM and through the vertex of contact with the ground. The net force and the net moment are both zero. Its CM will move with uniform motion and the direction of rotation will be constant (see Figure 5.1).

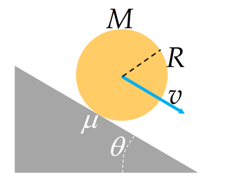


Fig. 5.5: Sphere moving down an inclined plane

In all these cases, and in accordance with what has been seen in Section 3.11, the possible displacements of a rigid rigid body's particles will be a subset of

$$d\vec{r}_i = d\vec{r}_C + d\vec{\varphi} \times \vec{r}_{i(C)} \tag{5.1}$$

The constraint we have to impose is keeping the axis of rotation constant. If $\hat{u}$ is the unit and constant vector in the direction of the axis of rotation, the possible rotational displacements will go from three degrees of freedom, represented by the vector character of $d\vec{\varphi}$, to one degree of freedom, represented by $d\varphi$ according to:

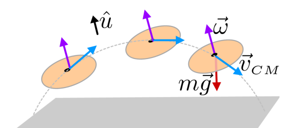


Fig. 5.6: A disc maintains the direction of rotation while the CM traces a parabola

$$d\vec{\varphi} = \hat{u}\ d\varphi \tag{5.2}$$

In this course, we will deal exclusively with the motions of the body in a plane. Valid relations in general will be indicated as **3D** and those that are valid only for plane motions will be indicated as **2D**. It should be noted that this chapter will use the nomenclature of the particle system seen in Chapter 3 although, for continuous bodies, the expressions in the form of summations are converted into integrals over the whole body (expression (3.1) in the Chapter 3 introduction).

5.1 Translation equation in 3D

According to equation (3.91) in Section 3.13, the equation of motion for the translation of a rigid body is

$$\vec{F} = \frac{d\vec{P}}{dt} = m\vec{a}_{CM} \tag{5.3}$$

where $\vec{F}$ is the resultant of the forces external to the body. The equation of the translational motion of the body is therefore that of the motion of its CM (Figure 5.7).

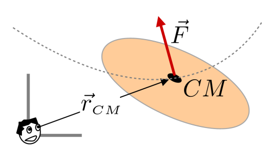


Fig. 5.7: Translational dynamics of a body

5.2 Rotation equation for (2D) plane motion. Moment of inertia

Since the motion is in a plane, the rotation of a rigid body is reduced to a single component in the direction of the rotational angular velocity vector $\vec{\omega}$, which is a fixed direction of space $\hat{u}$ and is perpendicular to the plane of motion. In Section 3.13, the general equation of the rotational dynamics of a rigid body was derived:

$$\frac{d\vec{L}_{(C)}}{dt} = \vec{M}_{(C)} \tag{5.4}$$

with

$$\vec{L}_{(C)} = \sum_{i=1}^{N} \vec{r}_{i(C)} \times m_i \vec{v}_i \tag{5.5}$$

$$\vec{M}_{(C)} = \sum_{a} \vec{r}_{a(C)} \times \vec{F}_a \tag{5.6}$$

where C can be either the centre of mass, CM, or a fixed point of the body, if it exists, and $\vec{r}_{a(C)}$ is the position vector of the point of application of the external force to the body $\vec{F}_a$ with respect to C. The rotation equation of the body in its plane of motion can be obtained by projecting the vector equation (5.4) onto the direction of its axis of rotation defined by its vector $\hat{u}$. This is so because the angular velocity can be written as $\vec{\omega} = \omega\hat{u}$, where ω is the component of the vector $\vec{\omega}$ in the direction of $\hat{u}$. ω is the only degree of freedom of rotation that the body has. In terms of the possible displacements, we have $d\vec{\varphi} = d\varphi\hat{u}$. Now, taking into account the general equation of dynamics for rotations, equation (3.94), we obtain, for rotations with axis $d\vec{\varphi} = d\varphi\hat{u}$:

$$\frac{d\vec{L}_{(C)}}{dt} \cdot \hat{u} = \vec{M}_{(C)} \cdot \hat{u} \tag{5.7}$$

Since $\hat{u}$ is a constant vector, (5.7) can be expressed as follows:

$$\frac{dL_{(C)}}{dt} = M_{(C)} \tag{5.8}$$

where

$$L_{(C)} = \vec{L}_{(C)} \cdot \hat{u} = \left(\sum_{i=1}^{N} \vec{r}_{i(C)} \times m_i \vec{v}_i \right) \cdot \hat{u} \tag{5.9}$$

$$M_{(C)} = \vec{M}_{(C)} \cdot \hat{u} = \left(\sum_{a} \vec{r}_{a(C)} \times \vec{F}_a \right) \cdot \hat{u} \tag{5.10}$$

are, respectively, the **angular momentum** and the **momentum of the forces with respect to the fixed axis of rotation** $\hat{u}$ passing through C.

According to (3.84), with $\vec{\omega} = \omega\hat{u}$, we have

$$\vec{v}_i = \vec{v}_C + \omega\ \hat{u} \times \vec{r}_{i(C)}$$

and substituting in $L_{(C)}$ of (5.9) we have

$$L_{(C)} = \left(\sum_{i} \vec{r}_{i(C)} \times m_i \left[\vec{v}_C + \omega\ \hat{u} \times \vec{r}_{i(C)} \right] \right) \cdot \hat{u}$$

and developing

$$L_{(C)} = \left(\sum_{i} \vec{r}_{i(C)} \times m_i \vec{v}_C \right) \cdot \hat{u} + \left(\sum_{i} m_i \vec{r}_{i(C)} \times \left[\ \hat{u} \times \vec{r}_{i(C)} \right] \right) \cdot \omega\hat{u}$$

The first term on the right is zero if C is the CM, since $\sum_i m_i\vec{r}_{i(CM)} = m\vec{r}_{CM(CM)} = 0$, or a **fixed point** on the body, since $\vec{v}_C = 0$. In these conditions,

$$L_{(C)} = \left(\sum_i m_i\vec{r}_{i(C)} \times [\,\hat{u} \times \vec{r}_{i(C)}]\right) \cdot \omega\hat{u}$$

Taking into account the vector relation

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{C} \times \vec{A}) \cdot \vec{B} \tag{5.11}$$

extracted from (1.13), $L_{(C)}$ can be expressed as

$$L_{(C)} = \vec{L}_{(C)} \cdot \hat{u} = \sum_{i=1}^{N} m_i\, r^2_{i(|C)}\omega$$

where $r_{i(|C)} = |\hat{u} \times \vec{r}_{i(C)}|$ is the distance between the particle m_i and the $\hat{u}$-axis passing through C (see Figure 5.8).

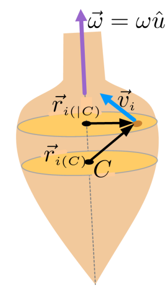


Fig. 5.8: Velocity of any particle of a rigid body in its plane motion

➔ We define the **moment of inertia of the body with respect to the axis of rotation passing through** C:

$$I_{(C)} = \sum_{i=1}^{N} m_i\, r^2_{i(|C)} \tag{5.12}$$

We can write the angular momentum with respect to the axis through C as

$$L_{(C)} = I_{(C)}\,\omega \tag{5.13}$$

so that the equation of the body for plane rotations becomes

$$\frac{dL_{(C)}}{dt} = M_{(C)} \tag{5.14}$$

In the general case, the forces external to the body, $\vec{F}_a$, can have any direction but only the components normal to the axis will contribute to the moment in the direction of the axis. Here we already assume that the forces $\vec{F}_a$ involved in the moment expression are normal to the axis. According to the relation (5.11), M_C can be expressed in the form

$$M_{(C)} = \vec{M}_{(C)} \cdot \hat{u} = \sum_a \left(\vec{r}_{a(C)} \times \vec{F}_a\right) \cdot \hat{u} = \sum_a (\hat{u} \times \vec{r}_{a(C)}) \cdot \vec{F}_a$$

However, since (see Figure 5.9) $\hat{u} \times \vec{r}_{a(C)} = \hat{u} \times \vec{r}_{a(|C)}$, $M_{(C)}$ can be written as

$$M_{(C)} = \sum_a (\hat{u} \times \vec{r}_{a(|C)}) \cdot \vec{F}_a$$

then, applying (5.11) again results in

$$M_{(C)} = \sum_a \left(\vec{r}_{a(|C)} \times \vec{F}_a\right) \cdot \hat{u} = \sum_a F_a d_a \tag{5.15}$$

where d_a is the distance of the line of action of the force $\vec{F}_a$ from the axis of rotation C.

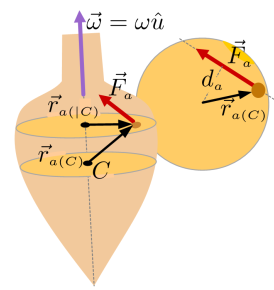


Fig. 5.9: An external force $\vec{F}_a$ acting on a rigid body

One aspect to be emphasised is that $I_{(C)}$ is invariant by translations and plane rotations of the body, i.e., it does not change its value because the body performs translations or plane rotations. This is an intrinsic characteristic of the body with respect to the axis of rotation passing through C (C being either a fixed point or the CM). $I_{(C)}$ could change if the body shape changed, i.e., if the body was no longer rigid.

Therefore, in the case of a rigid body, the moment of inertia about an axis $I_{(C)}$ does not change with time and the fundamental equation of rotation for a body in plane motion (5.14) can be written in the form

$$M_{(C)} = I_{(C)}\,\alpha \tag{5.16}$$

where $\alpha = \frac{d\omega}{dt}$ is the angular acceleration. The expression (5.16) has an analogy with the fundamental equation of translational dynamics $\vec{F} = m\vec{a}$: If the (inertial) mass m can be understood as a measure of the tendency to maintain the translational velocity, the moment of inertia $I_{(C)}$ can be understood as a measure of the tendency to maintain the angular velocity of rotation.

Some moments of inertia about an axis of homogeneous bodies

The table on page 339 shows the moments of inertia of well-known homogeneous rigid bodies about the indicated axes passing through their centres of mass.

Some properties of moments of inertia about an axis

The moment of inertia of a rigid body with respect to an axis has, among others, two important properties:

➔ ***Superposition of moments of inertia:*** *If we have two bodies 1 and 2 with moments of inertia about the same axis through C, $I_{(C)1}$ and $I_{(C)2}$, the moment of inertia $I_{(C)}$ of the composite body is*

$$I_{(C)} = I_{(C)1} + I_{(C)2} \tag{5.17}$$

This theorem is easily deduced from the definition of the moment of inertia about an axis as a summation or integral over the body. As in the case of the centre of mass of composite bodies, equation (3.7), we can also decompose a body into simple bodies (of known moment of inertia about the required axis) and find the moment of inertia of the composite body with (5.17). We may again consider bodies with holes by simply interpreting the addition of (5.17) as adding mass. If we want to remove mass, we must subtract.

➔ ***Steiner's theorem:*** *If a rigid body has moments of inertia $I_{(C)}$ and $I_{(CM)}$ about two parallel axes passing through C and CM respectively, then*

$$I_{(C)} = I_{(CM)} + md^2 \tag{5.18}$$

where m is the mass of the body and d is the distance between the two axes.

Proof. The moments of inertia $I_{(C)}$ and $I_{(CM)}$ of the body with respect to two parallel axes passing through, respectively, point C and the centre of mass CM (see Figure 5.10) are expressed in the form:

$$\begin{aligned} I_{(C)} &= \sum_{i=1}^{N} m_i\, r^2_{i(|C)} \\ I_{(CM)} &= \sum_{i=1}^{N} m_i\, r^2_{i(|CM)} \end{aligned}$$

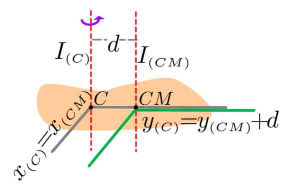


Fig. 5.10: Steiner's theorem

Looking at Figure 5.10, if $x_{i(CM)}$ and $y_{i(CM)}$ are the coordinates of a particle of the body of mass m_i with respect to CM, and $x_{i(C)}$ and $y_{i(C)}$ are the coordinates of the same particle with respect to C, then $x_{i(C)} = x_{i(CM)}$ and $y_{i(C)} = y_{i(CM)} + d$. Thus,

$$r^2_{i(|C)} = x^2_{i(C)} + y^2_{i(C)} = {x_{i(CM)}}^2 + (y_{i(CM)} + d)^2$$

Under these conditions, $I_{(C)}$ can be written as follows

$$I_{(C)} = \sum_{i=1}^{N} m_i \left(x^2_{i(CM)} + (y_{i(CM)} + d)^2\right)$$

that is to say,

$$I_{(C)} = \sum_{i=1}^{N} m_i({x_{i(CM)}}^2 + {y_{i(CM)}}^2) + 2\left(\sum_{i=1}^{N} m_i y_{i(CM)}\right) d + \left(\sum_{i=1}^{N} m_i\right) d^2$$

Now it is only necessary to note that, by the definition of the centre of mass, $\sum_{i=1}^{N} m_i y_{i(CM)} = 0$. The two terms that do not cancel are $I_{(CM)}$ and md^2. ■

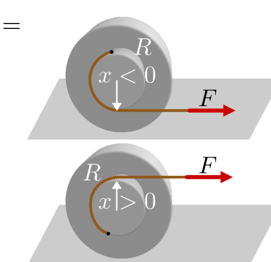


Figure for Problem 5.2.1

Problem 5.2.1. A wheel of radius R, mass m and moment of inertia I, has an axle of radius $|x|$ and is subjected to a constant horizontal force F acting by means of a coiled rope attached to the axle. As shown in the figure, we use $x > 0$ to indicate that the constant force F is applied above the centre and $x < 0$ to indicate that the constant force F is applied below. Calculate the acceleration of the wheel (without forgetting the sign of rotation) and the necessary friction force (also its direction) as a function of x, if the wheel is rolling without sliding.

Solution

The situation where the force F is applied above the centre ($x > 0$) is analysed. The results will also be valid for $x < 0$.

According to the scheme of forces acting on the body, the equations of motion can be written as follows

$$\begin{cases} F - F_f = ma \;\; ; \quad N - mg = 0 \\ Fx + F_f R = I\alpha \end{cases}$$

to which must be added the condition that the wheel rolls without slippage

$$v = \omega R \Rightarrow a = \alpha R$$

From the first translation equation, we obtain

$$F_f = F - ma$$

and substituting in the second, we have

$$Fx + FR - maR = \frac{I}{R}a$$

from which it follows that the acceleration a of the wheel is

$$a = \frac{F\,(R + x)}{mR + \frac{I}{R}} = \frac{F}{m}\frac{\left(1 + \frac{x}{R}\right)}{\frac{I}{mR^2} + 1}$$

and it is observed that it is positive, i.e., the wheel will move to the right in the same direction as the force F.

With regard to the frictional force, F_f, it is easy to deduce from the obtained value of a that

$$F_f = F\left(\frac{\frac{I}{mR^2} - \frac{x}{R}}{\frac{I}{mR^2} + 1}\right)$$

From this expression, it can be seen that the direction of F_f is determined by the sign of $\frac{I}{mR^2} - \frac{x}{R}$, so that if $I > mRx$, F_f goes against F; if $I < mRx$, F_f has the same direction as F, and if $I = mRx$, F_f is zero.

The situation where $x < 0$ can be analysed by simply considering this possibility in the expressions found for $x > 0$, i.e., by making the substitution $x = -|x|$. We have

$$a = \frac{F\,(R - |x|)}{mR + \frac{I}{R}} = \frac{F}{m}\frac{\left(1 - \frac{|x|}{R}\right)}{\frac{I}{mR^2} + 1}$$

and for the friction force:

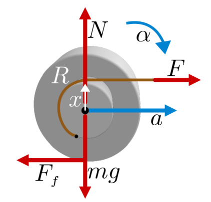
$$F_f = F\left(\frac{\frac{I}{mR^2} + \frac{|x|}{R}}{\frac{I}{mR^2} + 1}\right)$$

from which it follows that , being $|x| < R$, the body will move to the right and the frictional force will always be directed to the left.

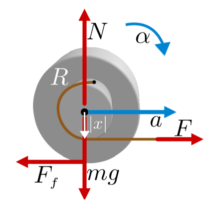
It is also possible to analyse the case where $|x| > R$.

If $x > 0$: a goes in the direction of F.

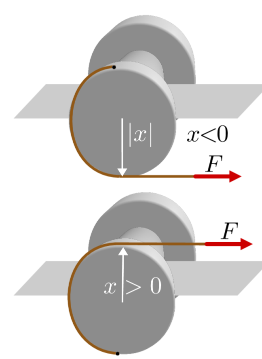
If $x < 0$: a goes in the opposite direction to F. ■



Solution to Problem 5.2.1, with $x > 0$



Solution to Problem 5.2.1, with $x < 0$



Solution to Problem 5.2.1, with $|x| > R$

Problem 5.2.2. A cubic block of mass m is at rest on a completely smooth floor when it receives a horizontal force $F = 3mg$ at a height $h = \frac{7}{8}b$. What is the value of the accelerations $\vec{a}$ of its centre of mass and of rotation α at the initial instant?

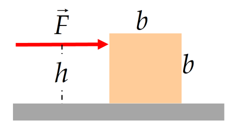


Figure for Problem 5.2.2

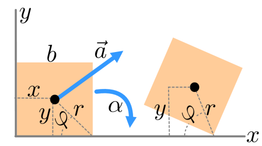
Solution

It is to be expected that the block will tend to rotate and its centre of mass will move in the direction shown in the figure, so that the CM of the cube will move in the $x-y$ plane with an acceleration $\vec{a}$ and will rotate around an axis perpendicular to the plane in the figure, which is passing through its CM with an angular acceleration α. The solution to the given situation is provided by the equation of translational dynamics (two components):

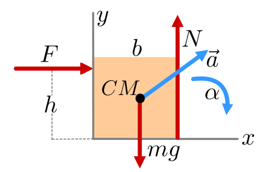
$$\vec{F} = m\vec{a}$$

and the equation of rotation (one component)

$$M_{(CM)} = I_{(CM)}\ \alpha$$



Solution to Problem 5.2.2



Solution to Problem 5.2.2

The distribution of forces acting on the body is shown in the second figure indicating the solution, in which we see that the normal due to the horizontal plane has been drawn at the right-hand vertex of the cube, since this is where it will act when the rotation starts. The two equations of motion for the translation of the centre of mass can be written as follows (where it is taken into account that $F = 3mg$):

$$3mg = ma_x \quad ; \quad N - mg = ma_y$$

For rotation, taking into account that, for a homogeneous cube of side b, the moment of inertia with respect to its CM is $I = \frac{1}{6}m\,b^2$, it follows that

$$3mg\left(\frac{7b}{8} - \frac{b}{2}\right) - N\frac{b}{2} = \frac{1}{6}m\,b^2\alpha$$

According to the above figures,

$$y = r\sin\varphi$$

Deriving respect to time, it turns out

$$\dot{y} = r\cos\varphi\ \ \dot{\varphi}$$

and deriving again and making $\ddot{y} = a_y$ and $\ddot{\varphi} = \alpha$, we have

$$a_y = -r\sin\varphi\ \ \dot{\varphi}^2 + r\cos\varphi\ \ \alpha$$

According to the problem conditions, before receiving the force F, and therefore, from the initial instant, $\dot{\varphi} = 0$. Also at this instant, $r\cos\varphi = \frac{b}{2}$. We obtain the ratio of accelerations a_y and α at the initial instant:

$$a_y = \frac{b}{2}\alpha$$

From the x-component of the equation of translation of the centre of mass it follows that

$$a_x = 3g$$

The y-component of the equation of translation of the centre of mass, at the initial instant, can be written as

$$N = mg + m\frac{b}{2}\alpha$$

and if we substitute this into the equation of rotation, it results in

$$3mg\left(\frac{7b}{8} - \frac{b}{2}\right) - \left(mg + m\frac{b}{2}\alpha\right)\frac{b}{2} = \frac{1}{6}m\,b^2\alpha$$

Isolating α, we have

$$\alpha = \frac{3g}{2b}$$

and, substituting the expression for a_y found above,

$$a_y = \frac{3g}{4}$$

At the beginning of the solution, we commented that "it is to be expected". Now we can corroborate that the solution found is the correct one due to the fact that it satisfies $N = \frac{7}{4}mg > 0$. If $N < 0$, the solution would not be correct, since $N < 0$ would indicate that there is something more than a contact with the ground. ■

Problem 5.2.3. Consider a robot similar to the one in the figure, with the following characteristics: total mass, $M = 85\,\text{kg}$; mass and moment of inertia of the wheels and the motor rotor, $m = 20\,\text{kg}$ and $I = 0.6\,\text{kg}\,\text{m}^2$; radius of the wheels, $R = 0.2\,\text{m}$; distance from the centre of the wheels to the robot's centre of mass, $d = 1\,\text{m}$.

Initially, the robot is at rest. To start in a straight line, the motor acts on the wheels by applying a torque (moment) $\mathcal{M} = 25\,\text{N}\,\text{m}$, which causes the robot to move forward. Assuming that the wheels do not slip, how much is the acceleration of the robot and what angle does it need to tilt to prevent it from falling?

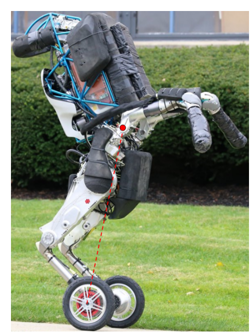
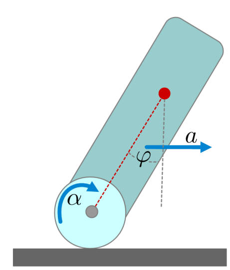


Figure for Problem 5.2.3

Solution

In the figure we can see a diagram of the robot as it moves forward with acceleration a. The assembly is not a rigid body, but it is made up of two parts that are rigid. The torque that drives the wheels is due to internal forces. For this reason, the two parts must be considered separately: the wheels with the part of the motor that rotates together with them (rotor); and the body of the robot, which does not rotate. The torque caused by the motor is $\mathcal{M}$, which, together with N' and F, constitute the internal forces in the contact between the two parts under consideration. The internal forces act on each part of the assembly in accordance with the action-reaction principle. $(M - m)g$ is the weight of the robot body and mg is the weight of the wheels. The normal N and the friction F_f are due to the contact of the wheel with the ground. The wheels do not slide, they roll without sliding, $\alpha R = a$.



Solution to Problem 5.2.3

We have the following motion equations:

(1) Horizontal translation of the wheels: $F_f - F = ma$

(2) Vertical translation of the wheels: $N - (N' + mg) = 0$

(3) Rotation of the wheels: $\mathcal{M} - F_f R = I\alpha$

(4) Horizontal translation of the body: $F = (M - m)a$

(5) Vertical translation of the body: $N' - (M - m)g = 0$

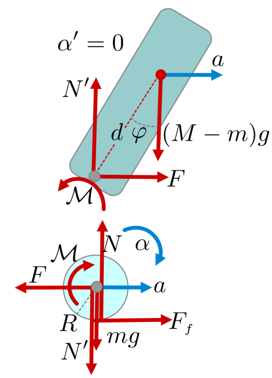
(6) Rotation of the body: $\mathcal{M} + F d \cos\varphi - N' d \sin\varphi = 0$

In order to solve our problem, we have enough with (1), (3), (4) and (6) and the condition of rolling without sliding. If we solve for F and F_f from (1) and (4), we find $F_f = Ma$. We can also obtain this equation by considering the entire robot (even though it is not a rigid body). Substituting in (3) with $\alpha = a/R$ and solving for a, we obtain:

$$a = \frac{\mathcal{M}\, R}{I + M\, R^2} = 1.25\,\mathrm{m/s^2}$$

Substituting F with the previous result in (6) and solving for φ, we obtain:

$$\sin\varphi = 0.165048 \Rightarrow \varphi = 0.165807\,\mathrm{rad} = 9.5^\circ$$ ■



Solution to Problem 5.2.3

5.3 Kinetic energy of rotation and translation. Energy conservation

In this section, the aim is to derive a very simple expression for the kinetic energy of a rigid body in plane motion. We start from the expression for the kinetic energy of a system of N particles (3.27):

$$E_c = \sum_{i=1}^{N} \frac{1}{2} m_i\, v_i^2 = \sum_{i=1}^{N} \frac{1}{2} m_i\, \vec{v}_i \cdot \vec{v}_i \tag{5.19}$$

According to (3.84), $\vec{v}_i = \vec{v}_C + \vec{\omega} \times \vec{r}_{i(C)}$, then (5.19) can be written as

$$E_c = \frac{1}{2} m v_C^2 + \vec{v}_C \cdot \left(\vec{\omega} \times \sum_{i=1}^{N} m_i\, \vec{r}_{i(C)} \right) + \sum_{i=1}^{N} \frac{1}{2} m_i\, \left(\vec{\omega} \times \vec{r}_{i(C)}\right)^2 \tag{5.20}$$

The first term is zero if C is a fixed point, since $\vec{v}_C = 0$; and it is the translational kinetic energy if $C = CM$. The second term is null in both of the following situations: if C is a fixed point, since $\vec{v}_C = 0$, and if $C = CM$, since $\sum_{i=1}^{N} m_i\, \vec{r}_{i(CM)} = 0$.

To deal with the last term, note that $\vec{\omega} \times \vec{r}_{i(C)} = \vec{\omega} \times \vec{r}_{i(|C)}$, since the component of $\vec{r}_{i(C)}$ parallel to $\vec{\omega}$ does not contribute to the vector product with $\vec{\omega}$.

Considering that $\vec{\omega}$ and $\vec{r}_{i(|C)}$ are perpendicular and, therefore, $|\vec{\omega} \times \vec{r}_{i(|C)}| = \omega r_{i(|C)}$, we have

$$\left(\vec{\omega} \times \vec{r}_{i(|C)}\right)^2 = \left(\vec{\omega} \times \vec{r}_{i(|C)}\right) \cdot \left(\vec{\omega} \times \vec{r}_{i(|C)}\right) = \omega^2 r_{i(|C)}^2 \tag{5.21}$$

Substituting (5.21) into (5.20) after cancelling the second term, we have

$$E_c = \frac{1}{2} m v_C^2 + \frac{1}{2} \sum_{i=1}^{N} m_i\, \vec{r}_{i(|C)}^{\,2}\, \omega^2 = \frac{1}{2} m v_C^2 + \frac{1}{2} I_{(C)}\, \omega^2 \tag{5.22}$$

If $C = CM$, then (5.22) is

$$E_c = \frac{1}{2}mv_{CM}^2 + \frac{1}{2}I_{(CM)}\,\omega^2 \tag{5.23}$$

and the kinetic energy of the body has two terms: the first one associated with the CM translation and the second one with the rotation around the CM.

If C is a fixed point, then (5.22) is

$$E_c = \frac{1}{2}I_{(C)}\,\omega^2 \tag{5.24}$$

and the kinetic energy of the body has only one term associated with the rotation around C.

The expressions (5.23, 5.24) are also valid for 3D situations, although in this case $I_{(CM)}$ (or $I_{(C)}$, if C is fixed) will not be constant, since the direction of the axis of rotation will not be fixed. For 2D situations, the direction is fixed and $I_{(CM)}$ (or $I_{(C)}$, if C is fixed) is constant.

Finally, it is not difficult to prove the corresponding conservation of mechanical energy theorem:

➔ ***Energy conservation***. *For a rigid body subjected to external forces that are either conservative with joint potential energy U or they are constraints with ideal reactions, its mechanical energy $E = E_c + U$ is conserved throughout the motion.*

In the case of rigid body systems whose forces have the same characteristics, as in the previous case, the energy will be the sum of the energies of each body and it will also be conserved throughout the motion.

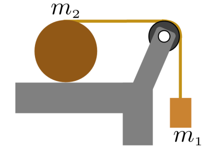


Figure for Problem 5.3.1

Problem 5.3.1. A mass of $m_1 = 1\,$kg hangs from the end of a rope of negligible weight passing through a frictionless pulley (see Figure). The rope is wrapped around a homogeneous cylinder of mass $m_2 = 8\,$kg and radius $R = 10\,$cm, which rotates without sliding in a horizontal plane. Find:

a) The acceleration of the mass m_1.

b) The rope tension.

c) The angular acceleration of the cylinder.

Solution

a) Bearing in mind that all the forces acting on the system are either conservative or they do not work in their displacement, the principle of conservation of mechanical energy E can be applied to it. This can be written as

$$E = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}I\omega^2 - m_1gy_1$$

where, being a solid cylinder, $I = \frac{1}{2}m_2R^2$.

Since the cylinder rolls without sliding on the horizontal surface and the rope does not slide,

$$v_2 = \omega R \;\; ; \;\; v_1 = 2\omega R \;\Rightarrow v_2 = \frac{v_1}{2}$$

and the time derivatives are

$$\alpha = \frac{a_1}{2R} \;\; ; \;\; a_2 = \frac{a_1}{2}$$

Let us apply conservation of energy by imposing that its time derivative is zero:

$$\dot{E} = 0 = (m_1 + \frac{m_2}{4} + \frac{I}{4R^2})v_1 a_1 - m_1 g v_1$$

from which it follows

$$a_1 = \frac{m_1 g}{m_1 + \frac{3m_2}{8}} = 2.45\,\text{m/s}^2$$

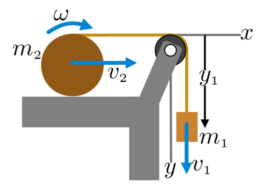
b) To determine the tension in the rope, the equation of translation can be applied to m_1

$$m_1 g - T = m_1 a_1 \Rightarrow T = m_1 g - m_1 a_1 = 7.35\,\text{N}$$

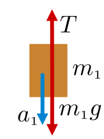
c) The angular acceleration of the cylinder is

$$\alpha = \frac{a_1}{2R} = 12.25\,\text{rad/s}^2$$

■



Solution to Problem 5.3.1



Solution to Problem 5.3.1

Problem 5.3.2. At a given instant, a homogeneous cylinder of mass M and radius R is left to roll from rest on the top of an inclined plane at an angle θ with the horizontal. Knowing that it rolls without sliding, determine:

a) The downward acceleration of the cylinder.

b) The velocity at the end of the plane after rolling over it for a distance L.

c) The minimum coefficient of friction μ between the cylinder and the plane that is compatible with rolling without slipping.

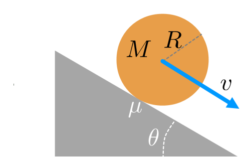
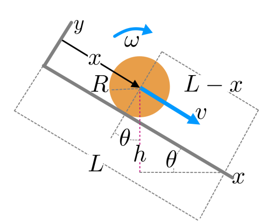


Figure for Problem 5.3.2

Solution

a) Since the forces acting on the cylinder are either conservative (its weight) or do not work during the motion (the normal of the plane and the friction force in the rolling displacement without sliding), the principle of conservation of mechanical energy E can be applied. Taking the base of the inclined plane as the origin of the gravitational potential energy, mechanical energy can be expressed as follows

$$E = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 + Mgh$$



Solution to Problem 5.3.2

From the figure, it can be seen that the height h of the centre of mass of the cylinder at any instant of its motion can be expressed as

$$h = (L - x)\sin\theta + R\cos\theta$$

Therefore, the mechanical energy of the cylinder can be written as

$$E = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 - Mgx\sin\theta + \text{constant}$$

and the nullity of its time derivative, using $\dot{x} = v = \omega R$, $\ddot{x} = a = \alpha R$ and $I = \frac{1}{2}MR^2$, where I is the moment of inertia of the cylinder with respect to the axis passing through its CM and perpendicular to the plane of the figure, gives us

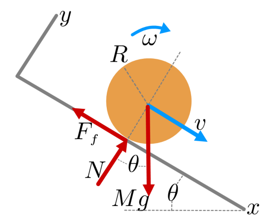
$$\dot{E} = 0 = \frac{3M}{2}va - Mgv\sin\theta$$

from which it follows

$$a = \frac{2}{3}g\sin\theta$$

b) The centre of mass of the cylinder describes a uniformly accelerated motion as it descends the inclined plane, with acceleration a and zero initial velocity. Therefore, the velocity it will reach at the end of the inclined plane, after having travelled a distance L, will be

$$v = \sqrt{2aL} = \sqrt{2\frac{2}{3}gL\sin\theta} = \sqrt{\frac{4}{3}gL\sin\theta}$$



Solution to Problem 5.3.2

c) This requires finding the frictional force acting between the cylinder and the inclined plane. If we take into account Newton's law for the translation in the normal direction and the rotation, we can write

$$N - Mg\cos\theta = 0$$
$$F_f R = I\alpha$$

from which we obtain

$$N = Mg\cos\theta$$
$$F_f = \frac{I\alpha}{R}$$

Now, the coefficient of friction μ is

$$\mu = \frac{F_f}{N}$$

if we substitute F_f, N, $I = \frac{1}{2}MR^2$ and $\alpha = \frac{a}{R}$, with the acceleration found in **a)**, we obtain

$$\mu = \frac{Ia}{R^2 Mg\cos\theta} = \frac{1}{3}\tan\theta$$ ■

Problem 5.3.3. A circular disk at rest with a radius of $0.5\,\text{m}$ and a moment of inertia of $4\,\text{kg}\,\text{m}^2$ can rotate about a fixed axis that passes through its centre and has a rope wound around its periphery. The rope is stretched with a constant force of $2\,\text{N}$ for $10\,\text{s}$. Assuming no friction, calculate the length of rope that becomes unwound within this time.

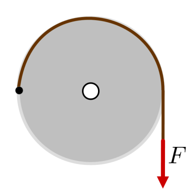
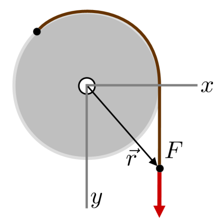


Figure for Problem 5.3.3

Solution

In addition to the force F, the disc is acted on by the weight and the reaction of the axis. Both have their point of application at the centre of the disc, which is a fixed point. Therefore, they do not work. The vertical force F applied at the periphery of the disc is constant; therefore, it is conservative. The mechanical energy of the disc, which consists of the kinetic and potential energies associated with F, will remain constant during its rotation. Taking into account that, at any given instant of the movement, the position vector of the point of application of the force F is $\vec{r} = (R, y)$ and, in addition, that $\vec{F} = (0, F)$, the potential energy associated with F is



Solution to Problem 5.3.3

$$U = -\vec{F} \cdot \vec{r} = -Fy$$

The mechanical energy E of the disc will be

$$E = \frac{1}{2} I\omega^2 - Fy$$

and the time derivative, taking into account that $\dot{y} = v$ and $v = \omega R$, will be

$$\dot{E} = I\omega\alpha - F\omega R$$

from which, with $\dot{E} = 0$, the angular acceleration of the disk is obtained

$$\alpha = \frac{FR}{I}$$

Finally, the acceleration a which lowers the point of application of the force F is also obtained:

$$a = \alpha R = \frac{FR^2}{I} = 0.125\,\mathrm{m/s^2}$$

Assuming that this point starts from rest and describes a uniformly accelerated rectilinear motion with acceleration a, the length L of the unwound rope for 10 s is

$$L = \frac{1}{2} a t^2 = 6.25\,\mathrm{m}$$ ■

Problem 5.3.4. Two masses of 1 and 2 kg are connected by an inextensible rope with no mass, which passes, without slipping, through a 1.35 kg cylindrical pulley with a fixed axis. Calculate the tensions of the rope.

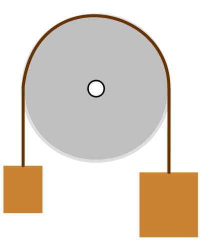


Figure for Problem 5.3.4

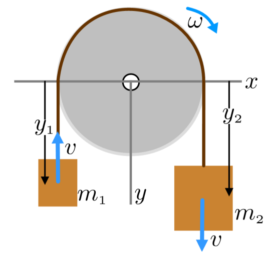
Solution

We take into account the following. First, the reaction forces of the pulley axis and its weight are forces whose application points are not displaced in the rotational motion. Second, $m_1 g$ and $m_2 g$ are conservative forces. Therefore, we can apply the principle of conservation of mechanical energy E of the system. This energy can be written in the following way (see the considered reference frame in the figure):

$$E = \frac{1}{2} m_1 v^2 - m_1 g y_1 + \frac{1}{2} m_2 v^2 - m_2 g y_2 + \frac{1}{2} I\omega^2$$

Under the problem conditions, $\dot{y}_1 = -v$, $\dot{y}_2 = v$, $v = \omega R$ and $a = \alpha R$, where ω and α are, respectively, the angular velocity and angular acceleration of the pulley, and

$I = \frac{1}{2}MR^2$ is the moment of inertia of the pulley with respect to the axis of rotation, which passes through its centre and is perpendicular to the plane of the figure (M is the mass of the pulley and R its radius).



Solution to Problem 5.3.4 $m_1 = 1\,\mathrm{kg}$, $m_2 = 2\,\mathrm{kg}$

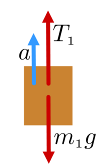
$$E = \frac{1}{2}(m_1 + m_2 + \frac{I}{R^2})v^2 - m_1 g y_1 - m_2 g y_2$$

Taking into account that E is conserved, its derivative will be zero:

$$\dot{E} = 0 = (m_1 + m_2 + \frac{I}{R^2})va - (m_2 - m_1)gv$$

From this last expression, substituting also I, the acceleration a can be found:

$$a = \frac{(m_2 - m_1)g}{m_1 + m_2 + \frac{M}{2}} = 2.67\,\mathrm{m/s^2}$$



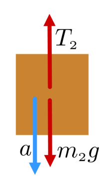
Solution to Problem 5.3.4. Free body diagram of m_1

Although equal in modulus, the accelerations of both bodies have opposite directions as vectors. In order to calculate the stresses in the rope, the dynamics of m_1 and m_2 must be analysed separately. With respect to m_1, taking into account that its motion is upward, its dynamics is reflected in the following equation:

$$T_1 - m_1 g = m_1 a$$

from which we obtain

$$T_1 = m_1(g + a) = 12.47\,\mathrm{N}$$



Solution to Problem 5.3.4. Free body diagram of m_2

As far as m_2 is concerned, taking into account that its motion is downward, its dynamics is reflected in the following equation:

$$m_2 g - T_2 = m_2 a$$

from which we obtain

$$T_2 = m_2(g - a) = 14.47\,\mathrm{N}$$

■

Problem 5.3.5. A wheel of radius 6 cm has an axis of radius 2 cm. The assembly has a moment of inertia of $0.004\,\mathrm{kg\,m^2}$ and a mass of 3 kg. The wheel rests on the ground and does not slip. Determine the direction of rotation of the wheel and its acceleration if, starting from rest, we pull horizontally on a rope wound around the axis with a force of 5 N, as shown in the figure.

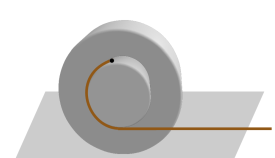


Figure for Problem 5.3.5

Solution

All the forces that act on the wheel are either conservative, or do not do any work in its rolling displacement without slipping. It is therefore a conservative system. Its mechanical energy E can be expressed in the form

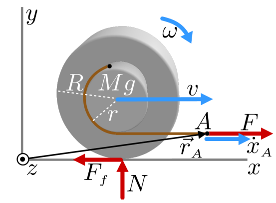
$$E = \frac{1}{2}Mv^2 + MgR + \frac{1}{2}I\omega^2 - \vec{F} \cdot \vec{r}_A + C$$

where $\vec{r}_A = (x_A, R)$ and, therefore, $U_F = -\vec{F} \cdot \vec{r}_A = -F x_A$ is the potential energy associated with the horizontally applied force $F = 2\,\mathrm{N}$, with C being a constant.

Since E is conserved, its time derivative will be zero:

$$\dot{E} = 0 = Mva + I\omega\alpha - F\dot{x}_A$$

where $\dot{x}_A$ is the velocity of the points on the rope, as A, $\vec{v}_A = (\dot{x}_A, 0, 0)$. If the rope does not slide on the axis, this is also the velocity at the point of contact between the wheel and the rope (see the figure for the solution). If we use $\vec{v}_A = \vec{v} + \vec{\omega} \times (\vec{r}_A - \vec{r})$, where, observing the figure, $\vec{r} = (x, R, 0)$ and $\vec{v} = (v, 0, 0)$ are, respectively, the position vector and the velocity of the cylinder's centre, and $\vec{\omega} = (0, 0, -\omega)$, we find



Solution to Problem 5.3.5

$$\dot{x}_A = v - \omega r$$

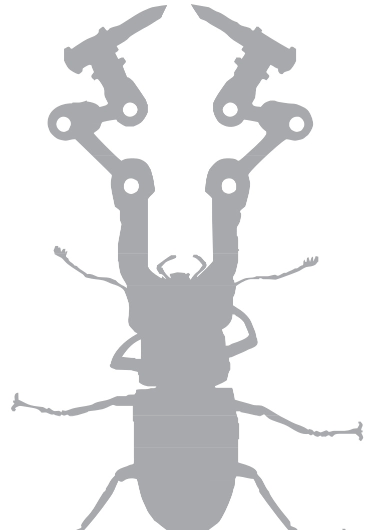
If this expression is substituted into the one for mechanical energy conservation, together with $v = \omega R$ and $a = \alpha R$, we find

$$\dot{E} = 0 = \left(M + \frac{I}{R^2}\right) va - F(1 - \frac{r}{R})v$$

and finally we get

$$a = \frac{F(1 - \frac{r}{R})}{M + \frac{I}{R^2}} = 0.81\,\mathrm{m/s}$$

$a > 0$ implies that $\alpha = \frac{a}{R} > 0$. Taking into account that we have taken the positive direction to the right and zero initial v, then, according to the figure, the wheel moves clockwise to the right. ■



→6

6 Small oscillations

Introduction

Oscillatory phenomena are very important. Everything that surrounds us tends to be in a position of stable equilibrium. The sea is in stable equilibrium and its waves are small oscillations around that equilibrium. A structure, such as a building, is in stable equilibrium and any perturbation that does not break it will also cause small oscillations around that equilibrium.

The basic approximation of solid matter's internal behaviour is rigidity, which is a configuration of stable equilibrium. The first approximation to the internal motion of solid matter is the small oscillations around its rigidity configuration.

The study of small oscillations is therefore a first approach to studying of the dynamic behaviour of many systems that at first sight appear to be immovable but, for whatever reason, then begin to *wobble*. It is quite remarkable that we can carry out this study without going into detail about the causes that alter the state of equilibrium.

6.1 Small oscillations around a stable equilibrium position

Let us consider a conservative system with one degree of freedom x of potential energy $U(x)$, which has a stable equilibrium position x_0. This means that $U(x)$ fulfils

$$\frac{dU}{dx}(x_0) = 0 \quad , \quad \frac{d^2U}{dx^2}(x_0) = k > 0 \tag{6.1}$$

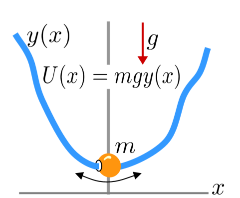


Fig. 6.1: A small ball strung on a wire performs harmonic oscillations around the stable equilibrium position

We redefine the reference frame so that the equilibrium position is $x_0 = 0$ and consider small deviations x around the equilibrium position $x_0 = 0$. We perform a Taylor serie of $U(x)$ and remain at the first significant order, i.e., at the first order where we get a non-zero result:

$$U(x) = U(0) + \frac{dU}{dx}(0)\ x + \frac{1}{2}\frac{d^2U}{dx^2}(0)\ x^2 + O[x^3] \approx \frac{1}{2}\,k\,x^2 + \text{constant} \tag{6.2}$$

Figure 6.1 provides an example that illustrates a generic situation. In a vertical plane, we give the desired shape, $y(x)$, to a guiding wire so that it has a minimum, where we place the origin of the reference frame $x = 0$. We string a small ball of mass m onto the wire and let it rest at the point where $x = 0$. We give it a little push and observe the motion it makes around $x = 0$. The potential energy will be

$$U(x) = mg\, y(x) \approx \frac{1}{2} k\, x^2 \quad ; \quad k = mg \frac{d^2 y}{dx^2}(0)$$

The kinetic energy will be

$$E_c = \frac{1}{2} m\, v^2 \;; \quad v = \sqrt{\dot{x}^2 + \dot{y}^2} = \dot{x} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

and, since near the equilibrium position $\frac{dy}{dx} \approx 0$, we have $E_c \approx \frac{1}{2} m\, \dot{x}^2$. The mechanical energy for small deviations from the equilibrium position turns out to be

$$E \approx \frac{1}{2} m\, \dot{x}^2 + \frac{1}{2} k\, x^2 \tag{6.3}$$

The equation of motion for small deviations from the equilibrium position will therefore be $m\, \ddot{x} = -k\, x$ and the system will carry out harmonic oscillations around this position. It is not only springs that oscillate harmonically, and we can in fact state that:

➔ *If any system with a stable equilibrium position starts close to this position with a small initial velocity, it will perform a harmonic motion.*

In this chapter, x is a position variable for the system, meaning it can be any length or angle. $x = 0$ is the stable equilibrium position around which the system oscillates. In these conditions, x is called ***elongation***.

6.2 Simple harmonic motion (SHM)

In Figure 6.2, we can see the landing gear of an aircraft. If the link between the wheel and the body of the aircraft were rigid, the contact with the runway would cause considerable damage. For this reason, an elastic element, such as a spring, is placed between the wheel and the aircraft body. As a result of the spring, when the aircraft body comes into contact with the runway, it starts a vertical harmonic oscillatory motion that does not stop. This is a **simple harmonic motion** (SHM). What we want to do is characterize the simple harmonic motion without focusing on the specific cause that originates it in each case. The vertical oscillatory motion of the plane is very similar to the horizontal motion of the ball in Figure 6.1, except that the former is caused by gravity and the guiding wire, while the latter is caused by the spring. We want a characterization of these cases, that allows us to treat them with the same expressions without having to tweak the formulas in order to

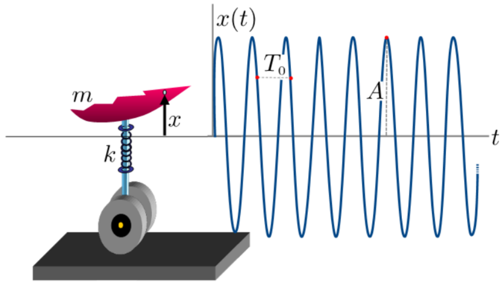


Fig. 6.2: The spring of an aircraft landing gear causes the aircraft to perform a harmonic oscillatory motion initiated on contact with the runway

accommodate each case in question. We can be guided by the motion equation $m\,\ddot{x} = -k\,x$.

SHM canonical equation

We say that a system with one degree of freedom x performs a simple harmonic motion if the equation of motion can be written in the canonical form:

$$\ddot{x} + \omega_0^2 x = 0 \tag{6.4}$$

where ω_0 is called ***free pulsation***.

General solution of the SHM

The general solution of the canonical equation of the SHM is

$$x(t) = A\sin(\omega_0 t + \varphi_0) \tag{6.5}$$

where A and φ_0 are integration constants.

The proof is immediate. Just derive the expression (6.5) twice and substitute x and $\ddot{x}$ into the canonical equation (6.4) to see that it is identically satisfied.

A and φ_0 are called, respectively, ***amplitude*** and ***initial phase*** and, as mentioned above, are integration constants. They can be fixed if we know the initial conditions for any time t_0, $x(t_0)$ and $\dot{x}(t_0)$. The argument of the trigonometric function sin, $\varphi(t) = \omega_0 t + \varphi_0$, is called the ***phase***. Thus, the initial phase is $\varphi_0 = \varphi(0)$.

From the point of view of their physical significance, we highlight the following concepts:

➔ **Free period**. This is the minimum time T_0 that must elapse for the same values of x and $\dot{x}$ to be repeated. That is, $x(t{+}T_0) = x(t)$ and $\dot{x}(t{+}T_0) = \dot{x}(t)$. This requires the phase to return to the same value (modulus 2π), i.e., $\varphi(t + T_0) = \varphi(t) + 2\pi$. We obtain $\omega_0 T_0 = 2\pi$.

Free frequency. This is the inverse of the free period.

$$f_0 = \frac{1}{T_0} = \frac{\omega_0}{2\pi} \tag{6.6}$$

Amplitude. This is the maximum value that the elongation $x(t)$ can reach. The amplitude is thus the factor $A > 0$ of the trigonometric function of the elongation $x(t)$ of (6.5).

Example of simple harmonic motion

An obvious and simple example where we can apply everything we have learned is that of a homogeneous sphere attached to the end of a spring, with the other end fixed and in the presence of gravity.

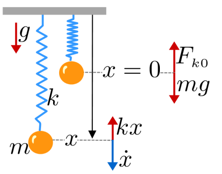


Fig. 6.3: A particle attached to a spring. Gravity affects the location of the equilibrium point, but not the motion relative to this point

We let the sphere *find* its own equilibrium position. We are not currently interested in where that is. Let us place the origin of the reference frame where the equilibrium position is, as shown in Figure 6.3. The x-coordinate is the elongation, since $x = 0$ is the equilibrium point. At this point, the acting forces are the weight mg and, in the opposite direction, the spring force, F_{k0}, which, being an equilibrium position, will have the same modulus as the weight, $F_{k0} = mg$.

We give an arbitrary small push (vertical translation) to the sphere. When it is at position x and moving at a speed $\dot{x}$, the acting forces are the weight and, in the opposite direction, the spring force F_k. The latter can be decomposed as $F_k = F_{k0} + kx$. Since $F_{k0} = mg$, the sum of the acting forces, $(mg - F_k)\hat{\imath}$, is $-kx\hat{\imath}$. Newton's equation of motion is therefore $m\ddot{x} = -kx$. Written in canonical form, it is

$$\ddot{x} + \frac{k}{m}x = 0 \tag{6.7}$$

Comparing this with the canonical equation of the SHM, we deduce that it is a simple harmonic motion, characterized by the free pulsation $\omega_0 = \sqrt{\frac{k}{m}}$. The period is, according to (6.6), $T_0 = 2\pi\sqrt{\frac{m}{k}}$. We note that gravity does not appear in the equation of motion. Gravity affects the location of the equilibrium point, but not the motion with respect to this point.

6.3 Damped harmonic motion (DHM)

We introduce a viscous damping force of type $-b\dot{x}$ into the landing gear of the aircraft (see Figure 6.4).

Now the oscillations of the aircraft's body become smaller and smaller and we call this a **damped harmonic motion (DHM)**.

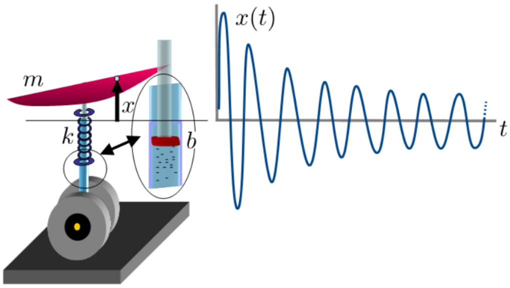


Fig. 6.4: The spring and damper in the landing gear of an aircraft cause the aircraft to make a damped harmonic oscillatory motion upon contact with the runway

It is important to note that, since the damping force is proportional to the velocity, the equilibrium position of the damped system is exactly the same as that of the system without the damping force.

We will try to characterize the damped harmonic motion without focusing on the specific cause in each situation. In this case, we shall use the equation of motion $m\,\ddot{x} = -b\dot{x} - k\,x$.

DHM canonical equation

We say that a system of one degree of freedom x performs a damped harmonic motion if the equation of motion can be written in the canonical form:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \tag{6.8}$$

where ω_0 is still called the *free pulsation* and γ is called the ***damping parameter***, which is related, as we shall see, to the damping force.

Often, we will have to distinguish between the damping parameter γ and the coefficient of friction b. To do this, let us note that, regardless of what x is, the units of γ are s^{-1}. The units of b must be such that bv has units of force, N. Thus, $[b] = \mathrm{N\,s\,m^{-1}} = \mathrm{kg\,s^{-1}}$.

General solution of DHM

The general solution of the canonical DHM equation has a different form depending on the relative values of ω_0 and γ.

Oscillating motion: $\omega_0 > \gamma$

The general solution can be expressed as

$$x(t) = A\ \mathrm{e}^{-\gamma t}\sin(\omega\,t + \varphi_0) \tag{6.9}$$

where A and φ_0 are integration constants, and

$$\omega = \sqrt{\omega_0^2 - \gamma^2} \tag{6.10}$$

is the **pulsation**.

The proof is immediate, although it is more complicated than in the case of SHM. One only has to derive expression (6.9) twice and substitute x, $\dot{x}$ and $\ddot{x}$ into the canonical equation (6.8) to see that, taking into account (6.10), it is identically satisfied.

A and φ_0 are called, respectively, the ***initial amplitude*** and ***initial phase*** and, as already mentioned, they are integration constants. They can be fixed if the initial conditions for any instant t_0, $x(t_0)$ and $\dot{x}(t_0)$ are known. The argument of the trigonometric sin function, $\varphi(t) = \omega t + \varphi_0$, is called ***phase***. Thus, the initial phase is $\varphi_0 = \varphi(0)$.

Now the analogous concepts of free period and amplitude cannot be exactly the same, although we can try to make a common generalisation of them.

- **Variable amplitude**. This is the factor $A(t) = A\,\mathrm{e}^{-\gamma t}$, with $A > 0$, of the trigonometric function of the elongation $x(t)$ of (6.9). The variable amplitude is the envelope of the elongation, which we can see in Figure 6.5, where the variable amplitude has a tangent contact near each local maximum of the elongation.

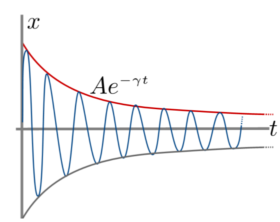


Fig. 6.5: The variable amplitude has a tangent contact near each local maximum of the elongation

- **Period**. The minimum time T that must elapse to repeat the same values of **elongation ratio** $\frac{x(t)}{A(t)}$ and **elongation velocity ratio** $\frac{\dot{x}(t)}{\omega A(t)}$. For this, as in the free case, it is necessary that the phase returns to the same value (modulus 2π), i.e., $\varphi(t+T) = \varphi(t) + 2\pi$. We obtain $\omega T = 2\pi$.

- **Frequency**. This is the inverse of the period.

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \tag{6.11}$$

Overdamped motion: $\omega_0 < \gamma$

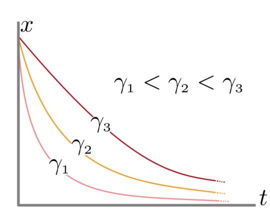


Fig. 6.6: Elongation plots for different values of γ in the overdamped case

In Figure 6.6, we can see that, because of the strong damping, the elongation does not oscillate. The oscillating form of the general solution is not adequate. Since $\omega_0 < \gamma$, ω becomes imaginary. The integration constants A and φ_0 would have to take imaginary values to make the elongation real. All this can be done and the general solution takes, in this case, the form

$$x(t) = C_1\,\mathrm{e}^{-\gamma_{(+)} t} + C_2\,\mathrm{e}^{-\gamma_{(-)} t} \tag{6.12}$$

where C_1 and C_2 are integration constants that can be fixed if we know the initial conditions and

$$\gamma_{(\pm)} = \gamma \pm \sqrt{\gamma^2 - \omega_0^2} \tag{6.13}$$

The proof is immediate. Just derive expression (6.12) twice and substitute x, $\dot{x}$ and $\ddot{x}$ into the canonical equation (6.8) to see that, taking into account (6.13), it is identically satisfied.

Critical motion: $\omega_0 = \gamma$

This is the case that separates the two previous ones. From the mathematical point of view, neither of the two previous solutions represents the critical motion well because, although there are no imaginary magnitudes, they contain only a constant of integration. It is easy to show that if $\omega_0 = \gamma$, the general solution of equation (6.8), with two integration constants, is

$$x(t) = (C_1 + C_2\, t)\, \mathrm{e}^{-\gamma t} \tag{6.14}$$

where C_1 and C_2 are integration constants that can be fixed if the initial conditions are known.

Example of damped harmonic motion

Continuing with the example of the homogeneous sphere attached to the end of a spring, with the other end fixed and in the presence of gravity, we now surround the sphere with a fluid that causes a viscous friction force of coefficient b and also an Archimedean push. As the sphere is homogeneous, the centre of buoyancy coincides with the centre of mass.

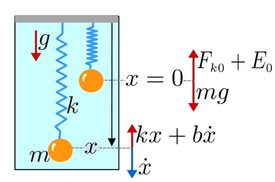


Fig. 6.7: Homogeneous sphere attached to a spring and in a fluid. The equilibrium point is independent of the viscous force. The net force does not depend on either gravity or buoyancy

We let the sphere *find* its equilibrium position which is where we establish the origin of the reference frame, as shown in Figure 6.7. The x-coordinate is the elongation, considering that $x = 0$ is the equilibrium point. At this point, the acting forces are the weight mg and, in the opposite direction, the spring force and the Archimedean buoyant force $F_{k0} + E_0$, which, being an equilibrium position, will have the same modulus as the weight when added together: $F_{k0} + E_0 = mg$.

We give an arbitrary push (vertical translation) to the sphere. When it is at position x, moving with velocity $\dot{x}$, the acting forces are the weight and, in the opposite direction, the spring force, the friction and the thrust $F_k + b\dot{x} + E_0$. Since $F_k = F_{k0} + kx$ and $F_{k0} + E_0 = mg$, we have that the sum of the acting forces, $(mg - E_0 - F_k - b\dot{x})\hat{\imath}$, is $(-kx - b\dot{x})\hat{\imath}$. Newton's equation of motion is thus $m\ddot{x} = -kx - b\dot{x}$. Written in canonical form, it is

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0 \tag{6.15}$$

Comparing this with the canonical equation of the DHM, we deduce that it is a damped harmonic motion characterised by the free pulsation $\omega_0 = \sqrt{\frac{k}{m}}$ and the damping parameter $\gamma = \frac{b}{2m}$. Depending on the relative values of ω_0 and γ, it will be an oscillatory, overdamped or critical motion.

Problem 6.3.1. A pulley with a moment of inertia I about its axis performs small damped oscillations while rotating without any friction on the axis: the rope has a negligible mass, always remains taut and never slides. A pair of equal springs of negligible mass and recovery constant k are fixed to the ground and to the ends of the rope. A disk of negligible mass is subjected to an aerodynamic friction force $-bv$, where v is the velocity of the disc and b is constant. For small oscillations, determine:

a) the equation of motion,

b) the trajectory equation

c) the period

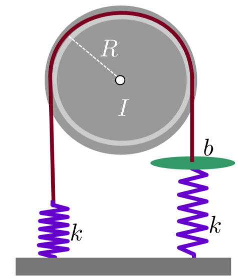
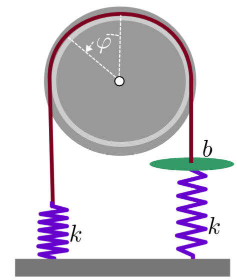


Figure for Problem 6.3.1



Solution to Problem 6.3.1

Solution

For the elongation, we take the angle φ, and, thus, at equilibrium $\varphi = 0$.

a) The equation of rotational motion for the pulley $M = I\alpha$ will be

$$I\ddot{\varphi} = -2kR\varphi R - bR\dot{\varphi}R$$

This is so because, when we are in equilibrium, the two springs have an equal and opposite moment $M_{Ek0} + M_{Dk0} = 0$; when we turn an angle φ in the positive direction (see the figure), the spring on the left has a smaller moment, since it has shortened $R\varphi$, $M_{Ek} = M_{Ek0} - k\ R\varphi\ R$, and the spring on the right has a larger moment, since it has lengthened $R\varphi$, $M_{Dk} = M_{Dk0} - k\ R\varphi\ R$. Thus, the moment resulting from the action of the springs is $M_{Dk} + M_{Ek} = -2k\ R\varphi\ R$. The moment of the viscous force $-bv = -bR\dot{\varphi}$ is $-bvR = -bR\dot{\varphi}R$.

We write the equation of motion in canonical form:

$$\ddot{\varphi} + \frac{b\ R^2}{I}\dot{\varphi} + \frac{2kR^2}{I}\varphi = 0$$

and, by comparison,

$$\gamma = \frac{bR^2}{2I} \quad ; \quad \omega_0 = \sqrt{\frac{2kR^2}{I}}$$

b) According to the problem statement, the trajectory correspons to an oscillating motion:

$$\varphi(t) = A_\varphi\ \mathrm{e}^{-\gamma t}\ \sin(\omega t + \xi_0) \quad ; \quad \omega = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\frac{R^2}{I}\left(2k - \frac{b^2\ R^2}{4I}\right)}$$

where A_φ is the angular amplitude and ξ_0 is the initial phase.

c) The period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{R^2}{I}\left(2k - \frac{b^2 R^2}{4I}\right)}}$$ ■

6.4 Forced harmonic motion (FHM)

We now assume that the aircraft runway in the previous sections contains a series dips that give it a sinusoidal shape (see Figure 6.8). The body of the aircraft thus receives a sinusoidal force that prolongs the motion in spite of the damping. What does this motion look like in the long run? What is the relationship between the spring, the dip frequency and the aircraft's vertical motion? We say that the system is forced or externally **excited**.

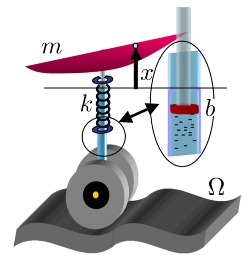


Fig. 6.8: The runway is not perfectly horizontal and flat; thus, its dips give it a sinusoidal shape. This means that the damped oscillatory motion of the aircraft does not stop and therefore receives a sinusoidal excitation force due to the dips in the runway maintaining the motion

Here, we try to characterize the forced harmonic motion without focusing on the specific cause that originates it in each case. In this case, we will be guided by the equation of motion $m\ddot{x} = -kx - b\dot{x} + F_E(t)$, where F_E is an external excitation force that we assume to be sinusoidal.

FHM canonical equation

We say that a system with one degree of freedom x performs a forced harmonic motion if the equation of motion can be written in the canonical form:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = B\sin(\Omega\, t + \theta_0) \tag{6.16}$$

where ω_0 is still called the *free pulsation*, γ the *damping parameter*, and the three new constants B, Ω and θ_0 are respectively related, as we will see, to the amplitude, frequency and initial phase of the excitation force (or torque).

General solution of the FHM

Since equation (6.16) is linear with respect to $x(t)$, the general solution can be written as the sum of the general solution of the homogeneous equation $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$, $x_h(t)$, which coincides with the equation of motion of the DHM, and a particular solution $x_p(t)$ of (6.16). $x(t) = x_h(t) + x_p(t)$. $x_h(t)$ is the corresponding solution of the DHM. In any of the three cases, the DHM is a damped motion that vanishes with time. This time depends on the damping parameter and can be evaluated as $t \gg \gamma^{-1}$, so that the factor $\mathrm{e}^{-\gamma t} \sim 0$. We say that $x_h(t)$ is the **transient** term. The particular solution x_p is found by trial and error. We start from

$$x_p(t) = A_p \sin(\Omega\, t + \theta_0 - \varphi_p) \tag{6.17}$$

where A_p and φ_p are constants that must be adjusted so that (6.17) effectively satisfies equation (6.16). If we substitute $x_p(t)$, $\dot{x}_p(t)$ and $\ddot{x}_p(t)$, found from (6.17), into (6.16), we get

$$\tan\varphi_p = \frac{2\gamma\Omega}{\omega_0^2 - \Omega^2} \quad ; \quad \sin\varphi_p \geq 0 \tag{6.18}$$

$$A_p = \frac{B}{\Omega\sqrt{4\gamma^2 + \left(\Omega - \frac{\omega_0^2}{\Omega}\right)^2}} \tag{6.19}$$

Therefore, the general solution of the FHM (6.16) can be written as a transient term x_h, general solution of the DHM, plus a **stationary** term x_p, defined according to (6.17) with a particular amplitude A_p according to (6.19) and an initial phase $\theta_0 - \varphi_p$, where φ_p is defined according to (6.18). φ_p is the phase difference between excitation and elongation.

For example, in the oscillatory case, $\omega_0 > \gamma$, the general solution is

$$x(t) = A\ \mathrm{e}^{-\gamma t}\sin(\omega\, t + \varphi_0) + A_p \sin(\Omega\, t + \theta_0 - \varphi_p) \tag{6.20}$$

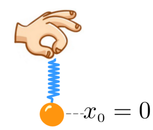


Fig. 6.9: Instead of fixing the upper end of the spring to the ceiling, we hold it by hand. If we keep our hand still, the equilibrium point is the same as before and does not move: $x_0 = 0$

where A and φ_0 are integration constants. At the beginning of the motion there is no definite pulsation. It is a superposition of two harmonic motions of respective pulsations ω and Ω. After a time, $t \gg \gamma^{-1}$, the transient term disappears and we are left with only the stationary term. The stationary term $x_p(t)$ is an oscillatory harmonic motion of pulsation Ω (determined by the exciter), of amplitude A_p and of initial phase $\theta_0 - \varphi_p$. None of these are related to the initial conditions. They are fixed by quantities present in the equation of motion and, thus, in the system: B, ω_0, γ, Ω and θ_0.

Example of forced harmonic motion

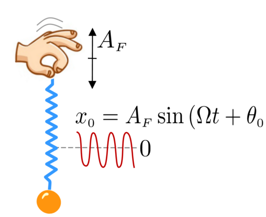


Fig. 6.10: If we move our hand vertically, with a sinusoidal motion of amplitude A_F and pulsation Ω, we will cause the equilibrium point to move around 0 according to $x_0 = A_F \sin(\Omega\, t + \theta_0)$

We continue with the example of the sphere tied to one end of a spring and under the same conditions as in the previous section. Now, however, we hold the other end of the spring by hand (see Figure 6.9).

With the hand still, the equation of motion will be exactly the same as in the case of the spring fixed to the ceiling, equation (6.15), which we can write in the form

$$m\ddot{x} = -b\dot{x} - k(x - x_0) \tag{6.21}$$

although it is now $x_0 = 0$.

If we move our hand vertically with a sinusoidal motion of amplitude A_F and pulsation Ω, we will cause the equilibrium point to move around $x = 0$ according to $x_0 = A_F \sin(\Omega\, t + \theta_0)$ (see Figure 6.10). This external excitation causes the spring force to change through the change in x_0. Substituting the new value of x_0 into (6.21), we get

$$m\,\ddot{x} = -b\,\dot{x} - k(\,x - A_F\sin(\Omega t + \theta_0))$$

which can be written as

$$m\,\ddot{x} = -b\,\dot{x} - kx + kA_F\sin(\Omega t + \theta_0)$$

where we see that the effect of moving the hand is the appearance of a force acting on the sphere $F = kA_F\sin(\Omega t + \theta_0)$. If we express it in canonical form

$$\ddot{x} + \frac{b}{m}\,\dot{x} + \frac{k}{m}x = \frac{kA_F}{m}\sin(\Omega t + \theta_0) \tag{6.22}$$

we obtain the same expressions for ω_0 and γ and also $B = \frac{kA_F}{m}$.

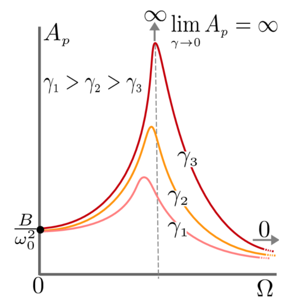


Fig. 6.11: Plots of $A_p(\Omega)$ for different values of the damping parameter $\gamma_1 > \gamma_2 > \gamma_3$. If $\omega_0^2 > 2\gamma^2$, there is a maximum. When $\gamma \to 0$, the maximum is at $\Omega = \omega_0$ and tends to infinity. When $\Omega \to \infty$, the result is $A_p(\infty) \to 0$. When $\Omega \to 0$, the system oscillates with amplitude $A_p = \frac{B}{\omega_0^2}$. In the latter case, if it were the sphere in the example, $B = \frac{kA_F}{m}$ and we obtain $A_p = A_F$, i.e., the sphere goes up and down without the spring deforming

Resonance phenomena

As we have already mentioned, the initial conditions do not determine the characteristics of the oscillating motion resulting from applying an exciter to a certain system but, instead, these characteristics are determined by the magnitudes of either the system or the exciter. In particular, amplitude A_p has a value determined by a certain combination of these quantities according to (6.19). We will study A_p as a function of Ω, since Ω is the quantity in (6.19) that does not belong to the system but to the exciter. The question is, given the system values of ω_0 and γ, is there a value of Ω that makes A_p particularly large? The plots of $A_p(\Omega)$ can be seen in Figure 6.11 for different values of γ. In general, whenever $\omega_0^2 > 2\gamma^2$, $A_p(\Omega)$ has a maximum. To which value of Ω does the maximum correspond? This is easy to answer, since we only have to derive $A_p(\Omega)$ with respect to Ω, equal to zero, and isolate Ω.

➔ **Amplitude resonance**. Whenever $\omega_o^2 > 2\gamma^2$, the excitation pulse Ω that causes the elongation amplitude A_p to have a maximum, which we call the ***amplitude resonance pulsation***, Ω_{RA}, is

$$\Omega_{RA} = \sqrt{\omega_0^2 - 2\gamma^2} \tag{6.23}$$

In addition to the amplitude resonance, the velocity resonance (i.e., the pusaltion Ω for which the velocity amplitude $\Omega A_p(\Omega)$ has a maximum value) is also of interest.

➔ **Velocity resonance**. The excitation pulse Ω that causes the velocity amplitude ΩA_p to have a maximum, which we call the ***velocity resonance pulsation*** Ω_{RV}, is

$$\Omega_{RV} = \omega_0 \tag{6.24}$$

It is a remarkable fact that, according to (6.18), the phase difference between excitation and elongation, φ_p, at velocity resonance is $\frac{\pi}{2}$.

Velocity resonance is closely related to energy issues, as we will see in the next subsection.

Energy and power

The expression of energy will depend on the particular system we are dealing with. However, knowing that, for the SHM, $\dot{E} = 0$ has to be the equation of motion, we can deduce that the energy of a harmonic motion can always be written as

$$E = \frac{1}{2} M_I \dot{x}^2 + \frac{1}{2} M_I \omega_0^2 x^2 \tag{6.25}$$

where M_I is a factor containing the inertia. For example, in the case of a translationally oscillating body, M_I is simply the mass, as in the example of the sphere attached to a spring discussed throughout this chapter.

If it is a DHM oscillating with $\gamma \ll \omega_0$, we can write the time evolution of the energy in the approximate form:

$$E = \frac{1}{2} M_I \omega_0^2 A(t)^2 = \mathrm{e}^{-2\gamma t} E(0) \tag{6.26}$$

In the case of an FHM, energy is not conserved due to the presence of friction and external excitation. In this case, we can write

$$\dot{E} = -2 M_I \gamma \dot{x}^2 + M_I B \sin\left(\Omega t + \theta_0\right) \, \dot{x} \tag{6.27}$$

since, if we develop $\dot{E}$ according to (6.25), (6.27) is the equation of the FHM motion.

When we are in the stationary phase, the average energy in a period, $\bar{E}$, is conserved, since if we calculate $\bar{E}$ as

$$\bar{E} = \frac{1}{T} \int_t^{t+T} E \; dt = \frac{1}{4} M_I A_p^2 \left(\omega_0^2 + \Omega^2\right) \tag{6.28}$$

we can see that $\bar{E}$ does not depend on t. There is frictional dissipation, but the average of the sum of the excitation and dissipation power cancels out.

Taking into account the exciter power term (i.e., the second term on the right-hand side of (6.27),), the average excitation power, $\bar{P}$, is

$$\bar{P} = \frac{1}{T}\int_{t}^{t+T} P dt = \frac{1}{T}\int_{t}^{t+T} M_I B \sin\left(\Omega t + \theta_0\right)\ \dot{x} dt$$
$$= \frac{1}{2} M_I B A_p \Omega \sin\varphi_p = \frac{1}{2} Z A_p^2 \Omega^2 \sin\varphi_p \tag{6.29}$$

where Z is called the ***mechanical impedance***:

$$Z = \frac{F_0}{A_p \Omega} = M_I \sqrt{4\gamma^2 + \left(\Omega - \frac{\omega_0^2}{\Omega}\right)^2} \tag{6.30}$$

and $F_0 = M_I B$.

As we can see, if the pulsation Ω coincides with the velocity resonance pulsation, $\Omega = \omega_0$, $A_p \Omega$ will be maximum and $\sin\varphi_p = 1$. Thus, we can say:

➔ **Maximum excitation power**. At velocity resonance, the average power developed by the exciter is maximum.

The relative difference between Ω_{RV} and Ω_{RA} is

$$\frac{\Omega_{RV} - \Omega_{RA}}{\Omega_{RV}} = \left(\frac{\gamma}{\omega_0}\right)^2 + O\left(\left(\frac{\gamma}{\omega_0}\right)^4\right) \tag{6.31}$$

For relatively small damping, $\gamma \ll \omega_0$, we will have $\Omega_{RV} = \Omega_{RA} \simeq \omega_0$. In these cases, it is only the free pulsation ω_0 that characterises the resonant behaviour of the system.

→7

7 Mechanical waves

Introduction

So far, we have studied matter in a rigid body model. Now we will consider the internal structure of this matter, for which we will take a block of homogeneous rigid material and look inside. There is no need to contemplate the atomic level, as it is enough to consider sufficiently large volumes whose amounts of matter remain constant in equilibrium. Bearing in mind that these portions interact with each other, we can construct a model by representing each volume of matter with a particle. The different particles are connected by springs which as we know from Chapter 6, and for small deviations from equilibrium, these springs adequately represent the interaction forces (see Figure 7.1)

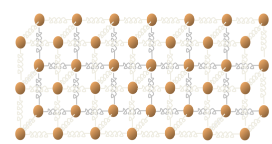


Fig. 7.1: Matter model beyond rigidity

Once we have our model constructed, we will see that suddenly displacing one particle from its equilibrium position will cause everything to start to vibrating internally. Stated plainly, we would say that a wave propagates through the medium (see Figure 7.2).

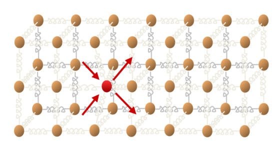


Fig. 7.2: The sudden displacement of a single particle will cause a wave to propagate through the medium

In this chapter, we will study how to deal with the internal movements of matter, that is, when it ceases to be rigid and becomes elastic.

7.1 Waves

If we locally **perturb** any of a medium's properties, the perturbation will be transmitted to the whole medium. The propagation of this disturbance is called a ***wave*** or ***wave motion***. Because these are mechanical disturbances to some of the material media's properties, the term of them is **mechanical waves**. When there is no confusion, we simply call them waves. The point or set of points where the initial disturbance occurs is called the ***focus***.

Let us think about some everyday experiences related to waves.

a) If we pull at a point on a taut string (see Figure 7.3), we will observe how the deformation produced is transmitted at a velocity v. If the string is tighter, the velocity of propagation is higher.

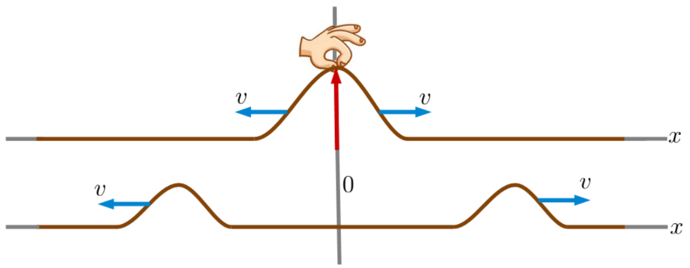


Fig. 7.3: Propagation of a perturbation along a taut string

b) If we hit a railway track with a hammer, after a short time, a person at a distance with his hand on the track will feel the vibration.

c) If we throw a stone into the water of a pond, we will see how the disturbance spreads circularly over the entire surface at a constant speed.

d) If we explode a firecracker, the rapid combustion of the gunpowder causes the displacement of gases that compress the spherical layer of air closest to the firecracker. Since air is very elastic, this initial compression expands the air, thus compressing the spherical layer that surrounds the first one, and so on in a succession of compression and expansion. The result is a sound wave that propagates spherically through the air.

All mechanical waves have the following common features:

- They occur in a **medium**, and one property of this medium is that it can be disturbed. We call this perturbation a ***wavefunction*** or ***field***. At equilibrium, the wave function is zero.
  - In the case of a taut string, the string defines the x-axis (see Figure 7.3). The wave function is the transverse displacement y of each point on the string relative to its equilibrium position, $y = 0$. Since each point on the string is determined by its position x, and the displacement y of each point depends on time, the wave on the entire string is characterised by the two-variable function, $y(x, t)$, which is the wave function of the string.
- In mechanical waves, the particles that make up the medium do not move on average, but only oscillate slightly around their equilibrium position. Waves do not transport matter, but they do propagate energy and momentum.

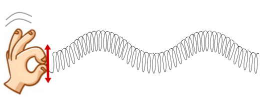


Fig. 7.4: A long spring is a good representation of a one-dimensional continuous medium model. We can generate a transverse wave by displacing some coils of the long spring transversely

If the oscillation velocity is perpendicular to the direction of propagation, we are dealing with ***transverse waves*** (see Figure 7.4).

If the oscillation velocity has the same direction as the propagation velocity, we are dealing with ***longitudinal waves*** (see Figure 7.5).

If the perturbation is a longitudinal or transverse displacement, it is called ***elongation***, as we have seen in Chapter 6.

If the initial disturbance is very short, the wave propagating in the medium is called a ***wave pulse***. This is the case in Figure 7.3.

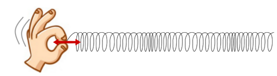


Fig. 7.5: Longitudinal wave caused by longitudinal displacement of some coils in the long spring

If the initial disturbance consists of an oscillation during a certain time interval Δt, what we have is a **wave train**. Figure 7.6, shows a harmonic wave train. If the interval Δt is very large, such as $\Delta t \rightarrow \infty$, the harmonic wave train will have become a **harmonic wave**. Harmonic waves, as we shall soon see, are a very important ideal case of waves.

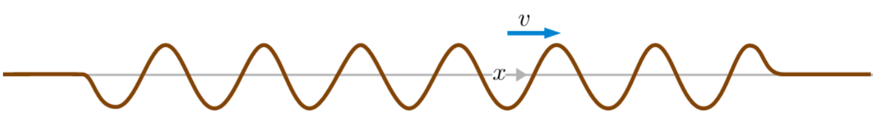


Fig. 7.6: A particular case of a wave train. If the initial disturbance consists of a harmonic oscillation which is prolonged for a time Δt, a harmonic wave train of length $v\Delta t$ is obtained

Wave fronts

A medium is called ***homogeneous*** if, given a fixed direction, the propagation speed is independent of the point under consideration.

A medium is ***isotropic*** if the propagation speed is the same in all directions, otherwise it is ***anisotropic***.

A medium is called ***non-dispersive*** if, at each point, all waves have the same speed.

A medium is called ***non-dissipative*** if the medium does not absorb the energy carried by the wave.

An ***ideal medium*** is a medium that is homogeneous, isotropic, non-dispersive and non-dissipative. In an ideal medium, the velocity of wave propagation is constant.

Regions of the medium connected by points with the same disturbance value are called ***wave fronts***. The points of a wave front are said to be ***in phase***. The direction of propagation is perpendicular to the wave fronts.

In an ideal medium, if the focus is a point or spherical, the wave front will be spherical. If the focus is flat, the wave front will also be flat. When it is far from the focus, a spherical wave front becomes approximately flat (see Figure 7.7). In an ideal medium, a **plane wave** or one-dimensional wave is caused by a flat focus. In practice, when a wave is far from a point or localised focus, it is a plane wave. In an ideal medium, a flat wave is characterized by having a single direction of propagation with constant velocity.

Waves on a taut string or on a long, narrow bar are examples of flat or one-dimensional waves, as they propagate in only one direction.

When the air in the atmosphere is contained within a range of a few metres and has no significant temperature gradients, it is an ideal medium for sound waves.

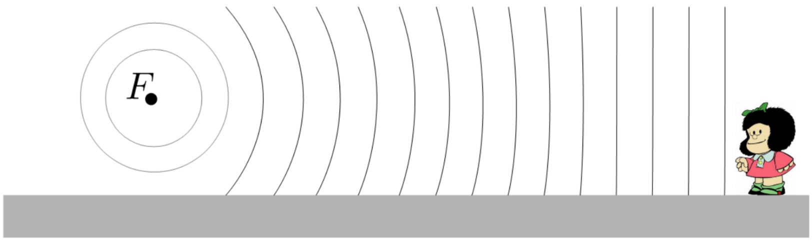


Fig. 7.7: A point source gives rise to spherical wave fronts. Away from the focus, these wave fronts are approximately flat and can be treated as plane waves

Wave fronts give a very clear picture of wave propagation. In fact, the study of waves began in the 17th century when Christiaan Huygens introduced the principle that bears his name.

→ ***Huygens' Principle***. *Any point on a wave front can become a new source that emits waves identical to those that originated it.*

Fig. 7.8: Christiaan Huygens (1629-1695) was a Dutch mathematician, physicist and astronomer

This principle makes it possible to explain and predict phenomena such as reflection, refraction and, above all, wave diffraction. Hyugens established his principle to demonstrate that light is some kind of wave, which contradicted Newton, who believed that light was nothing more than a beam of particles. By the 19th century, the statement of the principle was improved upon, first by Fresnell and then by Kirchhoff. Kirchhoff in fact showed that the principle was a consequence of the wave equation, which we will see in Section 7.2. Since this book focuses on studying waves from the mechanics point of view, we will not deal with Huygens' principle. Only at the end of chapter 8 will we discuss what diffraction is.

7.2 Plane waves and the wave equation

Plane waves: The wave function

Here, we will deal with plane waves in an ideal medium. For this type of medium:

a) The propagation speed v is a characteristic constant of the medium.

b) The perturbation retains its shape as it propagates (see Figure 7.9)

With these two conditions, a plane wave, described by the wave function y, which at equilibrium is $y = 0$, will be any y function of x and of t in the form

$$y(x,t) = y(x - vt) \qquad \text{or} \qquad y(x,t) = y(x + vt) \tag{7.1}$$

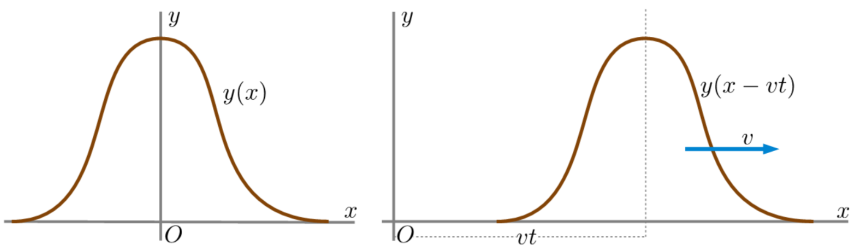


Fig. 7.9: The same perturbation at the initial time $t = 0$ and at a later time t in which it has propagated in the increasing x direction

Looking at Figure 7.9, we can see that $y(x - vt)$ satisfies conditions **a** and **b**. The wave fronts will be the points that have the same argument of the wave function, i.e., the points where $x - vt = C$, where C is a constant. Thus, for any instant t, the wave fronts are of the type $x = C + vt =$ constant. For each t, they are planes, and hence the name plane waves (see Figure 7.10). These planes travel at the velocity $\dot{x} = v$. Thus, $y = y(x - vt)$ represents a perturbation propagating in the increasing direction of the x-axis at velocity v. $y = y(x + vt)$ represents a perturbation propagating in the decreasing direction of the x-axis at velocity v.

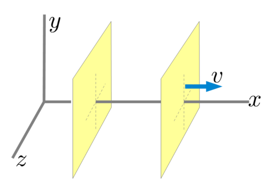


Fig. 7.10: The wave fronts of a plane wave are planes parallel to one another

It is necessary to distinguish between the oscillation and velocity of propagation. If the wave is plane and travels in the direction of the x-axis, it will have the generic form $y(x,t)$.

- The **oscillation velocity**, v_{osc}, is

$$v_{\text{osc}}(x,t) = \frac{\partial y(x,t)}{\partial t} \tag{7.2}$$

- The **propagation speed**, v_{prop}, for an ideal medium is the velocity $v_{\text{prop}} = |\dot{x}|$, such that $y = y\big(x(t), t\big)$ is constant. Thus:

$$\frac{dy}{dt} = 0 = \frac{\partial y}{\partial x}\dot{x} + \frac{\partial y}{\partial t}$$

from which follows

$$v_{\text{prop}} = |\dot{x}| = \left| \frac{\frac{\partial y}{\partial t}}{\frac{\partial y}{\partial x}} \right| \tag{7.3}$$

If the wave is transverse, $\vec{v}_{\text{osc}} \perp \vec{v}_{\text{prop}}$, and if it is longitudinal, $\vec{v}_{\text{osc}} || \vec{v}_{\text{prop}}$. If the medium is ideal, we know that the plane wave has the form $y(x,t) = y(x \pm vt)$ and the propagation speed is $v_{\text{prop}} = v$.

Plane waves: wave equation

If we take into account the two possible directions of propagation, a plane wave can have the form

$$y(x,t) = y(x - \varepsilon vt) \qquad \text{where} \qquad \varepsilon = \pm 1 \tag{7.4}$$

All possible plane waves in a medium can be characterised by a differential equation with $y(x,t)$ as the unknown. The equation can depend on v, which is a characteristic of the medium, but not on ε.

To find this equation, eliminating ε, we must derive (7.4). Now, $y(x,t) = Y(u)$, where $u = x + \varepsilon vt$. Applying the chain rule:

$$\frac{\partial y}{\partial x} = \frac{dY}{du}\frac{\partial u}{\partial x} = \frac{dY}{du} \qquad \frac{\partial^2 y}{\partial x^2} = \frac{d^2Y}{du^2}\frac{\partial u}{\partial x} = \frac{d^2Y}{du^2}$$

$$\frac{\partial y}{\partial t} = \frac{dY}{du}\frac{\partial u}{\partial t} = \frac{dY}{du}\varepsilon v \qquad \frac{\partial^2 y}{\partial t^2} = \frac{d^2Y}{du^2}\frac{\partial u}{\partial t} = \frac{d^2Y}{du^2}\varepsilon^2 v^2$$

Since $\varepsilon = \pm 1$, $\varepsilon^2 = 1$, we eliminate ε with the second derivatives. Eliminating $\frac{d^2Y}{du^2}$, we find the plane **wave equation**:

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0 \tag{7.5}$$

which is the wave equation propagating in only one direction, the x-direction. For waves propagating in different directions, that is, non-planar wave fronts, we must consider the three directions of space: x, y and z. Calling the wave function ψ and in following with similar reasoning as that for one dimension, the wave equation is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = 0 \tag{7.6}$$

Principle of superposition

The wave equation (7.5) is a **linear** partial derivative equation. The proof is simple: let $y_1(x,t)$ and $y_2(x,t)$ be two waves, i.e., two functions satisfying the wave equation

$$\frac{\partial^2 y_1}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y_1}{\partial t^2} = 0 \qquad \frac{\partial^2 y_2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y_2}{\partial t^2} = 0$$

Any linear combination of the two, $ay_1 + by_2$, with arbitrary constants a and b, satisfies the wave equation (7.5):

$$\begin{aligned}\frac{\partial^2 (ay_1 + by_2)}{\partial x^2} &- \frac{1}{c^2}\frac{\partial^2 (ay_1 + by_2)}{\partial t^2} \\ &= a\left[\frac{\partial^2 y_1}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y_1}{\partial t^2}\right] + b\left[\frac{\partial^2 y_2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y_2}{\partial t^2}\right] = 0\end{aligned}$$

The fact that the wave equation is linear means that it is in accordance with the **principle of superposition**, which greatly simplifies the behaviour and study of waves.

➔ ***The waves superposition principle***. *If more than one wave propagates through a medium, each wave propagates unaffected by the others.*

Since $y_1(x,t) = f(x - vt)$ and $y_2(x,t) = g(x + vt)$ are solutions of the wave equation, applying the principle of superposition gives us the **general solution of the wave equation**

$$y(x,t) = f(x - vt) + g(x + vt) \tag{7.7}$$

where f and g are arbitrary functions.

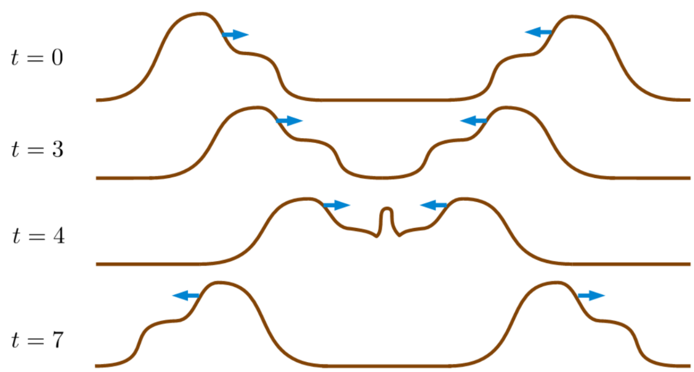


Fig. 7.11: If two pulses propagate in opposite directions on a string, the elongation of each point on the string will be the sum of the elongations that would be produced by each of the pulses separately

In Figure 7.11, we can see a case that illustrates the principle of superposition.

Problem 7.2.1. At the instant $t = 0$, a long, taut chain is struck, which initiates the propagation of a plane wave (wave pulse) that can be expressed as

$$y(x,t) = \frac{1}{12 + (x + 14t)^2} \tag{1}$$

where x and y are in m and the time t is in s. The profile of $y(x,t)$ is similar to the one shown in Figure 7.9.

a) Prove that this function is a plane wave and find the speed at which it propagates. What is the maximum elongation that each small link in the chain will experience as the wave propagates?

b) If $y(x,t)$ is a pulse, what approximate amplitude can we say it has?

c) What is the value of the oscillation velocity of any point x in the chain?

d) At the instant $t = 0$, which points will have the maximum oscillation velocity?

Solution

a) Indeed, the $y(x,t)$ function in (1) can be expressed as: $y = f(x - vt)$; therefore, it is a wave propagating at the velocity of $14\,\mathrm{m/s}$ in the direction of decreasing x. We can also see this by checking that it satisfies the wave equation (7.5).

For a given x, the maximum elongation y will be when the denominator of (1) is minimal: $x + 14t = 0$. Thus, for each x, the maximum y will be: $y_{\text{max}} = 1/12 = 0.0833$ m.

b) For an estimated value, we take as the pulse width the distance between the points on the string having an elongation $y = 1/10\, y_{\text{max}}$. Since the shape of the pulse does not change, we can work at $t = 0$:

$$\frac{1}{10}\,\frac{1}{12} \approx \frac{1}{12 + x^2} \qquad \text{from where} \qquad x \approx \pm 10 \text{ m}$$

Therefore, the pulse width is of the order of 20 m.

c) To find the oscillation velocity, we have to derive y:

$$v_{\text{osc}} = \frac{\partial y}{\partial t} = \frac{-28(x+14t)}{\left[12 + (x+14t)^2\right]^2}$$

d) For a given t, the point x of the pulse that will have the maximum oscillation velocity is given by

$$\frac{\partial v_{\text{osc}}}{\partial x} = 0 = \frac{-28 \cdot 14\left[12 + (x+14t)^2\right] + 112(x+14t)^2}{\left[12 + (x+14t)^2\right]^3}$$

from which $(x + 14t)^2 = 4$. If $t = 0$, then $x = \pm 2$ m. Given that the shape of the pulse is maintained, the two points of maximum oscillation velocity are 2 m from the maximum of the pulse. ■

Harmonic waves

In Figure 7.12, the rod of a vibrator describes a simple harmonic motion (SHM) of ω pulsation. An indefinitely long, taut string has one end attached to the rod. The movement is transmitted to the string and its propagation is a wave. If there is no dissipation, the wave will cover a very long stretch of the string after a certain time, such that all the points on the string far from the end will perform a SHM. Since the SHM of the rod does not stop, all points on the string perform a SHM at all times.

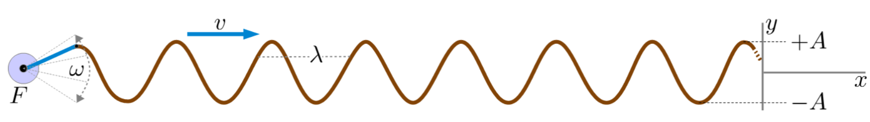


Fig. 7.12: The rod describes a simple harmonic motion (SHM) propagating along the taut string fixed at its end

The wave in Figure 7.12 is a type of plane wave called a ***harmonic wave***. A harmonic wave propagating in the increasing x-direction is a function of the form

$y(x,t) = f(x - vt)$ in which each point x performs a SHM, the pulsation of which is ω. These requirements impose the general form

$$y(x,t) = A\sin(kx - \omega t + \phi) \tag{7.8}$$

where A, k, ω and ϕ are constants with $\frac{\omega}{k} = v$. Thus, we will have

$$y(x,t) = A\sin(k(x - vt) + \phi) \tag{7.9}$$

and $y(x,t) = f(x - vt)$ is satisfied.

The parameters characterizing the harmonic wave are:

- **Amplitude** A. This is the quantity that provides the physical dimensions to y. If it is a displacement wave, A has length dimensions. If it is a pressure wave, A has pressure dimensions.

- **Period** T (s in SI units). If we film a point on the string, located horizontally at $x = x_0$, then $y(t) = y(x_0, t)$ is the vertical motion of this point. In Figure 7.13, $y(t)$ is plotted. T is the time period of the wave, or simply the **period**.

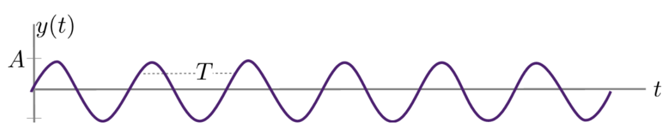


Fig. 7.13: Simple harmonic motion performed by perturbation y at point x_0 over time t

- **Pulsation or angular frequency** ω (rad/s in SI units). We can find the relationship to the period by taking into account the definition of the latter:

$$kx - \omega(t + T) + \phi = kx - \omega t + \phi - 2\pi$$

from which we find

$$\omega = \frac{2\pi}{T}$$

- **Frequency** f (s^{-1} = Hz in SI units, Hz= hertz). $f = 1/T$

- **Wavelength** λ (m in SI units). If we stop time, $t = t_0$, then $y(x) = y(x, t_0)$ becomes something like a photograph of a wave. Figure 7.14 represents $y(x)$ with λ being the wave's spatial period.

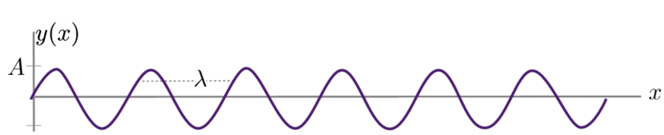


Fig. 7.14: Form of the wave function y at t_0 along the x coordinate of the medium

- **Wave number** k (rad/m in SI units). We can find the relationship between k and the wavelength by taking into account the definition of the latter:

$$k(x+\lambda)-\omega t+\phi = kx-\omega t+\phi+2\pi$$

from where we find

$$k=\frac{2\pi}{\lambda}$$

- **Propagation speed or phase speed** v (m/s in SI units). We can find the relationship to ω and k by taking into account that the phase has to be a function of $x-vt$:

$$kx-\omega t+\phi = k(x-\frac{\omega}{k}t)+\phi = k(x-vt)+\phi$$

thus

$$v=\frac{\omega}{k}=\frac{\lambda}{T}=\lambda f$$

- **Phase** $\varphi(x,t)$ (rad in SI units). This is the argument of the sine function in (7.8): $\varphi(x,t)=kx-\omega t+\phi$. It is measured in rad, never in degrees. Thus, ϕ is the **initial phase**, i.e., at $t=0$ and the origin $x=0$. Points with the same phase have the same state of motion.

  A **phase shift** is an increment of the phase $\Delta\varphi$:

  – Between two x points at the same instant t it is calculated as $\Delta\varphi=\varphi(x_2,t)-\varphi(x_1,t)=k(x_2-x_1)$. If $x_2=x_1+n\lambda$ with integer n, these two points have the same phase modulus 2π.

  – Between two instants t at the same point x it is $\Delta\varphi=\varphi(x,t_2)-\varphi(x,t_1)=\omega(t_2-t_1)$.

  With the appropriate value of ϕ, the same wave (7.8) can be written in several ways. For example,

$$y(x,t)=A\cos(kx-\omega t+\phi')=A\sin(\omega t-kx+\phi'')$$

  To evaluate phase shifts between two harmonic functions, both must be expressed in sine form or both in cosine form. In this text, we will opt for sin.

If the wave propagates in the opposite direction, i.e., in the direction that x decrease as in $y=f(x+vt)$, we will have $y(x,t)=A\sin(kx+\omega t+\phi)$.

Problem 7.2.2. The following harmonic wave propagates along a string

$$y(x,t)=3\times10^{-3}\cos\left[200\pi\left(t-\frac{x}{20}\right)\right] \qquad (1)$$

where all magnitudes are in SI units. For this wave, determine the following values.

a) The amplitude, period, frequency, wavelength and phase velocity.

b) The maximum transverse velocity and acceleration at any point on the string.

c) The distance between two consecutive points separated by a phase difference of $\pi/3$ rad.

Solution

To make (1) look like (7.8), it can be written as

$$y = 3 \times 10^{-3} \sin\left(10\pi x - 200\pi t \ + \frac{\pi}{2}\right) = A \sin\left(kx - \omega t + \frac{\pi}{2}\right) \qquad (2)$$

a) Now, comparing (2) with (7.8), we obtain:

Amplitude:	$A = 3 \times 10^{-3}$ m
Wave number:	$k = 10\pi = 31.42$ rad/m
Angular frequency:	$\omega = 200\pi = 628.3$ rad/s
Wavelength:	$\lambda = \frac{2\pi}{k} = \frac{1}{5} = 0.20$ m
Period:	$T = \frac{2\pi}{\omega} = \frac{1}{100} = 0.010$ s
Phase velocity:	$c = \frac{\omega}{k} = \frac{\lambda}{T} = 20$ m/s

b) All the points on the string perform a SHM in the direction of the y-axis. The velocity and acceleration are given by

$$v_{\text{osc}} = \frac{\partial y}{\partial t} = -\omega A \cos(kx - \omega t + \phi)$$

$$a_{\text{osc}} = \frac{\partial^2 y}{\partial t^2} = -\omega^2 A \sin(kx - \omega t + \phi) = -\omega^2 y(x, t)$$

The maximums are

- Maximum velocity: $v_{\text{max}} = \left(\frac{\partial y}{\partial t}\right)_{\text{max}} = \omega A = 0.6\pi = 1.885\,\text{m/s}$
- Maximum acceleration: $a_{\text{max}} = \left(\frac{\partial^2 y}{\partial t^2}\right)_{\text{max}} = \omega^2 A = 120\pi^2 = 1184\,\text{m/s}^2$

c) The phase difference is $\Delta\varphi = k\Delta x$. Thus, if $\Delta\varphi = \pi/3$ rad, we have

$$\Delta x = \frac{\Delta\varphi}{k} = \lambda \frac{\Delta\varphi}{2\pi} = \frac{1}{30} = 0.0333\,\text{m} \qquad \blacksquare$$

7.3 From Newton's laws to the wave equation

Newtonian mechanics predicts the existence of waves in elastic media and provides the expression of the propagation speed as a function of the properties of the medium.

Waves on a taut string

Let there be a very flexible taut string of linear density μ (kg/m in SI units), along which a transverse wave propagates. The position of the string in equilibrium defines the x-axis. Figure 7.15 shows the string differential between points 1 and 2

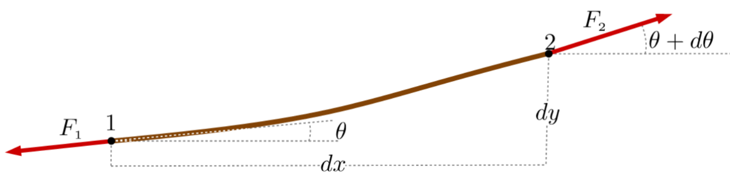


Fig. 7.15: A string differential

in motion. It also shows the two stresses $\vec{F}_1$ and $\vec{F}_2$ to which the string differential is subjected.

Let us apply Newton's second law to this string differential, which we will treat as a particle and we will restrict to small oscillations and large stresses. Consequently:

- $\theta \ll 1$ and $\sin\theta \approx \tan\theta \approx \theta$
- The string differential has only a transversely appreciable movement
- $dm = \mu\, dx$.
- The stresses $\vec{F}_1$ and $\vec{F}_2$ have the same modulus: $F_1 \approx F_2 \approx F$.
- The stressed string is horizontal in equilibrium. The weight force can be neglected.

With these assumptions, the sum of the transverse forces acting on $dm = \mu dx$ is

$$\begin{aligned} F_y &= F_{1_y} + F_{2_y} = -F\sin\theta + F\sin(\theta + d\theta) \approx \\ &\approx F(-\theta + \theta + d\theta) = F\, d\theta \end{aligned}$$

Since the slope of the differential of the string is the derivative

$$\tan\theta \approx \theta = \frac{dy}{dx} = \frac{\partial y}{\partial x}$$

we have

$$d\theta = \frac{d\theta}{dx}\, dx = \frac{\partial^2 y}{\partial x^2}\, dx$$

On the other hand, the transverse acceleration is $a_y = \frac{\partial^2 y}{\partial t^2}$. Newton's second law, $F_y = dm\, a_y$, is

$$F\frac{\partial^2 y}{\partial x^2}\, dx = \mu dx \frac{\partial^2 y}{\partial t^2} \tag{7.10}$$

which results in the wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} = 0 \tag{7.11}$$

From this, we deduce by comparison with (7.5) that the wave propagation speed is

$$v = \sqrt{\frac{F}{\mu}} \tag{7.12}$$

If we introduce section S of the string into the expression (7.12), we obtain

$$v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{F/S}{\mu/S}} = \sqrt{\frac{\tau}{\rho}} \tag{7.13}$$

where ρ is the volume density. We see again that the propagation speed is related to the characteristics of the medium according to $\sqrt{\frac{\text{stress}}{\text{density}}}$.

Problem 7.3.1. A long straight steel wire of radius $0.50\,\text{mm}$ is suspended from the ceiling. If a body of mass $10\,\text{kg}$ is suspended from its free end, determine the velocity of the pulse that propagates through the wire if we move any point slightly and transversely, then release it.
Datum. The density of steel is $\rho_{\text{steel}} = 7.80 \times 10^3\ \text{kg/m}^3$.

Solution
We apply the formula (7.12), taking as thread stress: $F = mg = (10\,\text{kg})\times(9.81\,\text{m/s}^2) = 98.1\,\text{N}$.
The linear density of the wire is

$$\mu = \frac{m}{\ell} = \frac{\rho\pi r^2\,\ell}{\ell} = \rho\pi r^2 = 6.123 \times 10^{-3}\ \text{kg/m}$$

Therefore, the velocity is $v = \sqrt{\dfrac{F}{\mu}} = 126.5\,\text{m/s}$ ■

Simple solid rod model

Let there be a very long, straight chain of N identical particle-springs connected to each other, as illustrated in Figure 7.16. The springs have a recovery constant k and the particles have a mass m. h is the distance between particles in equilibrium. Let us consider any mass, i, initially in equilibrium at the x_i position. The previous mass is at x_{i-1} and the posterior mass at x_{i+1}.

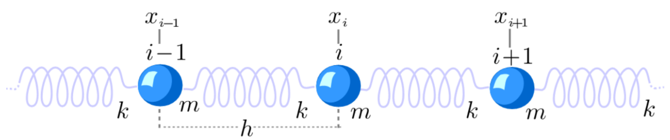


Fig. 7.16: Simple model of a solid rod in equilibrium

When a wave passes through this chain, the positions of the masses change. The i mass will move s_i.

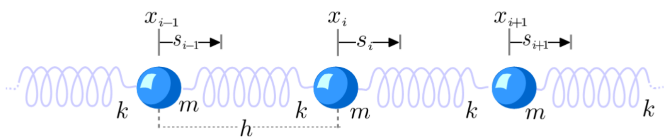


Fig. 7.17: Simple model of a solid rod in equilibrium when a wave propagates longitudinally

Looking at Figure 7.17, we can write the equation of motion for the i particle by applying Newton's second law:

$$m\,\frac{d^2 s_i}{dt^2} = +k(s_{i+1} - s_i) - k(s_i - s_{i-1})$$

which, multiplying and dividing by h, we can rewrite as

$$\frac{m}{h}\,\frac{d^2 s_i}{dt^2} = kh\left\{\frac{\left(\frac{s_{i+1}-s_i}{h}\right) - \left(\frac{s_i - s_{i-1}}{h}\right)}{h}\right\} \tag{7.14}$$

If h is very small and N is very large, we can make the transition to a continuous medium: keeping the length of the chain, $L = Nh$, finite, we make $N \to \infty$, while $h \to 0$. Instead of thinking that the particle i in equilibrium position x_i is displaced s_i, we will consider the particle in equilibrium position x is displaced s. In this case, s becomes a continuous function of x and time t, $s(x,t)$. Thus, we can approximate the right-hand member of (7.14) by means of partial derivatives:

$$\lim_{h\to 0}\left\{\frac{\left(\frac{s_{i+1}-s_i}{h}\right) - \left(\frac{s_i - s_{i-1}}{h}\right)}{h}\right\} = \lim_{h\to 0}\frac{\left(\frac{\partial s}{\partial x}\right)_x - \left(\frac{\partial s}{\partial x}\right)_{x+h}}{h} = \frac{\partial^2 s}{\partial x^2} \tag{7.15}$$

On the other hand, the ordinary derivative of the left-hand member of (7.14) can be understood as the partial derivative, because s_i is now $s(x,t)$. Therefore, from (7.15), (7.14) ends up looking like the wave equation

$$\frac{\partial^2 s}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 s}{\partial t^2} = 0 \qquad \text{with} \qquad v = h\sqrt{\frac{k}{m}} \tag{7.16}$$

That is to say, in the spring-mass chain, a longitudinal perturbation propagates from one mass to the next with a velocity v given by (7.16).

As mentioned above, (7.16) is correct if the length h is very small compared to the wavelength of the harmonic wave propagating in the chain. If the propagating wave is not a harmonic wave but a wave pulse, h must be much smaller than the pulse width.

It is not necessary for h to be the natural length of the spring. If it is, we can assume that, in equilibrium, the chain is not subject to any external tension. If it is not, we can assume that, in equilibrium, the chain is under constant external tension or compression.

Although this is a model, we can rewrite the propagation speed (7.16) by introducing the S-section of the chain, as

$$v = h\sqrt{\frac{k}{m}} = \sqrt{\frac{hk/S}{m/(hS)}} = \sqrt{\frac{\tau_1}{\rho}} \tag{7.17}$$

where $\rho = \frac{m}{hS}$ is the average volume density of the chain and $\tau_1 = \frac{hk}{S}$ is the stress that makes the deformation $\Delta L = L$. The corresponding stress is $F = \tau_1 S$.

Longitudinal and transverse waves on a solid rod

Consider a long, homogeneous bar of length L, uniform straight section S and volume density ρ (see Figure 7.18). The bar has a linear elastic behaviour due to longitudinal tension/compression. This means that, if a tensile (positive, outward) or compressive (negative, inward) force F is applied at each end of the bar, the bar will lengthen or shorten, respectively, by a small length ΔL, proportional to the stress F/S

$$\frac{\Delta L}{L} = \frac{1}{Y}\frac{F}{S} \tag{7.18}$$

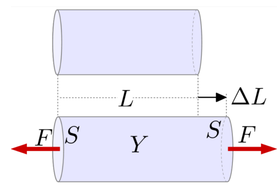


Fig. 7.18: Homogeneous long bar, of length L, uniform straight section S and volume density ρ, subjected to a tensile force F

where Y is **Young's modulus**, which characterises the elasticity of the bar and has dimensions of pressure that are usually measured in GPa $= 10^9$ Pa.

We can treat the bar following the spring-mass model that we have studied above. In particular, we can use (7.17) for the velocity of the longitudinal waves in the bar. The stress τ_1 is found from (7.18), with $\Delta L = L$, i.e. $\tau_1 = Y$. The propagation speed of longitudinal waves in the bar, v_{long}, will be as follows:

$$v_{long} = \sqrt{\frac{Y}{\rho}} \tag{7.19}$$

Table 7.1 gives the Young's and shear modulus, density and velocity of the longitudinal waves, v_{long}, for rods of various materials.

	Y	G	ρ	v_{long}	v_{tran}	Z
	($\times 10^9$ Pa)	($\times 10^9$ Pa)	(kg/m^3)	(m/s)	(m/s)	(rayl)
Steel	200	78	7800	5064	3160	39.5×10^6
Aluminium	70	26	2700	5092	3100	13.7×10^6
Copper	130	49	8900	3821	2350	34.0×10^6
Glass	~60	~24	~2500	~4900	~3100	$\sim12.3\times10^6$

Table 7.1: Young's and shear modulus, density and longitudinal and transverse wave velocities for rods made of several materials. Z is the specific acoustic impedance defined in the following chapter

The velocity (7.19) is valid for thin and long rods. In large solids, the velocity of longitudinal waves is given by this expression and multiplied by a factor that takes into account other elastic characteristics of the materials. Thus, using steel as an example, the correction factor is 1.15 and the resulting velocity is 5824 m/s.

Transverse waves also propagate in solids. In this case, the propagation depends on the **shear or torsional modulus**, G, which relates the torque of a two forces applied to the faces of one slice of a bar to the deformation characterised by an angle θ (see Figure 7.19).

$$\theta = \frac{1}{G}\frac{F}{S}$$

The velocity of the transverse waves, v_{tran}, which we give without proof, is

$$v_{tran} = \sqrt{\frac{G}{\rho}} \tag{7.20}$$

which also has the form $\sqrt{\frac{\text{stress}}{\text{density}}}$. Values of G for some materials are given in Table 7.1. It can be shown that, for any material, $G < Y$. Thus

➔ *For the same material, the velocity of longitudinal waves is greater than that of transverse waves.*

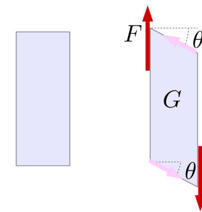


Fig. 7.19: The modulus G relates the torque applied to a *slice of bar* to the deformation characterised by θ

In fluids, $G = 0$ and the consequence is:

➔ *Within fluids, only longitudinal waves propagate.*

Earthquakes are the manifestation on the Earth's surface of **seismic waves** generated when large blocks of rock collide or move at a point in the Earth's crust. The seismic waves that are created are longitudinal, transverse and other types. They start from a focus, the hypocentre, and propagate as spherical waves. We know that longitudinal waves are faster (the average speed of these waves is well known) than transverse waves and are easily detected because they arrive earlier at seismic observatories. With three observatories recording the earthquakes, the position of the hypocentre can be determined by triangulation.

Longitudinal waves in a fluid

Let there be a longitudinal plane wave propagating in the direction of the x-axis through a fluid. Consider a tube of fluid of constant straight section S whose longitude is in the direction of propagation. In equilibrium the tube has a uniform density ρ. The wave can be described as either a **displacement wave** or as a **density wave**.

In the first case, the perturbation is the displacement, s, of each straight section of the tube. Thus, the section initially at x becomes $x + s$, where s will depend on x and t: $s(x, t)$.

In Figure 7.20 (a), the tube and a differential (or slice) of the tube bounded by the sections x_1 and $x_2 = x_1 + dx$ are shown in equilibrium before the wave passes through. The volume of this differential is $S\,dx$ and the mass is $dm = \rho\, S\, dx$.

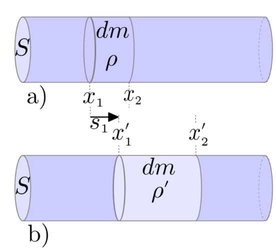


Fig. 7.20: Tube of fluid in the direction of propagation

Figure 7.20 (b) shows the same tube, but when the wave is passing through. The straight sections that bound the mass dm at positions x_1 and x_2 have become $x_1' = x_1 + s_1$ and $x_2' = x_2 + s_2$, where $s_2 = s_1 + ds$. The volume of this new differential slice is now $S\,(dx + ds)$.

Since the volume occupied by the mass dm has changed, the density in the new volume differential must also have changed from that of equilibrium ρ to $\rho + \rho_p$.

The mass dm can be expressed as

$$dm = \ \rho\, S\, dx = (\rho + \rho_p)\, S\, (dx + ds)$$

from which it follows

$$\rho_p = -(\rho + \rho_p)\frac{ds}{dx}$$

If we restrict ourselves to the study of small perturbations, $\rho_p \ll \rho$ and therefore $\rho + \rho_p \approx \rho$. Since s is a function of x and of t, we can write ds/dx as a partial derivative.We thus obtain the relationship between the density wave and the displacement wave:

$$\rho_p = -\rho\,\frac{\partial s}{\partial x} \tag{7.21}$$

A ρ_p density perturbation causes a pressure perturbation p or **acoustic pressure**:

$$p \approx dp = \left(\frac{\partial p}{\partial \rho}\right)_C d\rho \approx \left(\frac{\partial p}{\partial \rho}\right)_C \rho_p$$

and taking into account (7.21)

$$p = -\rho\left(\frac{\partial p}{\partial \rho}\right)_C \frac{\partial s}{\partial x}$$

i.e.,

$$p = -B\,\frac{\partial s}{\partial x} \tag{7.22}$$

where B is a quantity specific to the fluid, called the ***compressibility modulus***, which, for small variations in pressure or density, is constant. The subscript C in the above derivative indicates the compression process. B has pressure dimensions and is measured in Pa. Thus, we have

$$B = \rho\left(\frac{\partial p}{\partial \rho}\right)_C \qquad \text{or also} \qquad B = -V\left(\frac{\partial p}{\partial V}\right)_C \tag{7.23}$$

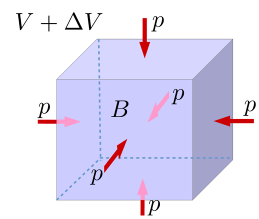


Fig. 7.21: Compressibility modulus B

We can write (7.23) for small volume $\partial V \approx \Delta V$ and pressure $\partial p \approx p$ variations (see Figure 7.21)

$$\frac{\Delta V}{V} = -\frac{1}{B}\,p \tag{7.24}$$

Figure 7.22 shows a long fluid cylinder of straight section S. More specifically, the “slice” of length dx and mass dm is shown. The left-hand side of the slice, located at x, is subjected to pressure $p(x)$ and the right-hand side, located at $x + dx$, is subjected to pressure $p(x + dx)$. The total force on the slice, $dF = -F' + F$, is

$$dF = \big[-p(x + dx) + p(x)\big]\,S = -S\,dp \tag{7.25}$$

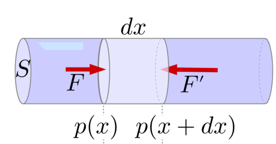


Fig. 7.22: A slice of fluid tube of length dx and mass dm

Let us apply Newton's second law to the slice, $dF = dm\, a$

$$dm\, a = dm\, \frac{\partial^2 s}{\partial t^2} = \rho\, S\, dx\, \frac{\partial^2 s}{\partial t^2}$$

and considering (7.25) and (7.22), we have

$$dF = -S\, dp = -S\, B\, \frac{\partial^2 s}{\partial x^2}\, dx$$

from which it follows

$$\frac{\partial^2 s}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 s}{\partial t^2} = 0 \qquad \text{with} \qquad v = \sqrt{\frac{B}{\rho}} \tag{7.26}$$

which is the equation for longitudinal waves in a fluid.

So far, we have not specified whether the fluid is a liquid or a gas. This is necessary for calculating (7.23), since the derivative $(dp/d\rho)_C$ depends on the compression process C. In the case of liquids, the process is purely mechanical[1] and the modulus B is constant. In Table 7.2, we have the modulus B and the density ρ of some liquids, as well as the velocity v of the corresponding waves.

[1] A process is purely mechanical, in relation to thermodynamics, when there is no variation in temperature or heat transfer; that is, when the process is isothermal and adiabatic

Liquid	B ($\times 10^9$ Pa)	ρ (kg/m³)	v (m/s)	Z (rayl)
Water	2.1	1000	1450	1.45×10^6
Glycerin	4.8	1261	1950	2.46×10^6
Mercury	27	13600	1400	19.0×10^6

Table 7.2: Compressibility modulus, density and wave velocity in liquids. Z is the specific acoustic impedance defined in the following chapter

Sound waves in a gas

When a gas is compressed, the density increases, the temperature changes and heat is transported. The two most important compression and expansion processes for a gas are **isothermal** and **adiabatic**. According to thermodynamics, the isothermal compressions of an ideal gas (for which $pV = $ const.) lead to a heat transport that keeps the temperature constant. In adiabatic compressions, there is no heat transport. Rapid processes are usually well represented by adiabatic processes.

In sound waves, compressions and expansions are so fast and gases are such poor conductors of heat that they can be considered adiabatic processes. The relationship between pressure and volume in the case of an ideal gas is $pV^\gamma = $ const., and between pressure and density it is $p/\rho^\gamma = $ const., where γ is the **adiabatic coefficient** of the gas. If the gas is monoatomic, $\gamma = 5/3$ and, if it is diatomic, $\gamma = 7/5$. Air can be considered diatomic.

For adiabatic processes in an ideal gas and taking into account $p/\rho^\gamma = $ const., the modulus B of (7.23) is

$$B = \rho\left(\frac{dp}{d\rho}\right)_{\text{adiab}} = \gamma\, p \tag{7.27}$$

Thus, from (7.26) and (7.27), the propagation speed of a sound wave in an ideal gas is

$$v = \sqrt{\frac{\gamma p}{\rho}} \tag{7.28}$$

According to the equation of state for ideal gases, $p = \frac{\rho}{M} RT$, where M is the molar mass of the gas, in kg, T is the absolute temperature, in kelvin, K; and R is the universal gas constant, $R = 8.314\,\mathrm{Jmol^{-1}K^{-1}}$, the speed of sound in an ideal gas can be written as

$$v = \sqrt{\frac{\gamma RT}{M}} \tag{7.29}$$

Given the gas, the speed depends only on the temperature.

Table 7.3 gives the values of the speed of sound for some ideal gases at $15°\mathrm{C}$ and the density of the gas at $1\,\mathrm{atm} = 1.013 \times 10^5\,\mathrm{Pa}$.

Gas	ρ (kg/m^3)	v (m/s)	Z (rayl)
Helium, He	0.169	916	155
Hydrogen, H_2	0.0845	1295	109
Air	1.23	340	418

Table 7.3: Density and speed of sound in gases, at $15°\mathrm{C} = 288.2\,\mathrm{K}$. Z is the specific acoustic impedance defined in the following chapter

7.4 Fourier analysis and synthesis

The principle of superposition tells us the following. If we have two plane harmonic waves in a medium of propagation speed v,

$$y_1(x,t) = A_1 \sin(\omega_1 t - k_1 x + \varphi_{10}) \qquad y_2(x,t) = A_2 \sin(\omega_2 t - k_2 x + \varphi_{20})$$

where ω_1, k_1, ω_2, k_2 must satisfy $v = \frac{\omega_1}{k_1} = \frac{\omega_2}{k_2}$, the superposition of both waves, $y_1 + y_2$, is also a propagating wave in this medium.

If $f(u)$ is a periodic function of u, of period T, then $f(u+T) = f(u)$, for any u. **Fourier's theorem** tells us that any periodic function of u, $f(u)$, of period T, can be decomposed as a sum of harmonic functions [2]

$$f(u) = \sum_{n=1}^{\infty} A_n \sin(\omega_n u + \varphi_n) \tag{7.30}$$

where the frequencies ω_n are integer multiples of the **fundamental frequency** $\omega_1 = \frac{2\pi}{T}$

$$\omega_n = n\,\omega_1 \qquad n = 1,\, 2,\, 3,\, \ldots$$

and the function

$$f_1 = A_1 \sin(\omega_1 u + \varphi_1)$$

[2] If the function is not periodic, a decomposition into harmonic functions similar to (7.30) can also be done; but the difference is that, instead of a summation for $\omega_n = n\omega_1$, we will have an integral and ω becomes a continuous value

is the **fundamental harmonic**. The other terms, $A_2 \sin(\omega_2 u + \varphi_2)$, etc., are the **higher harmonics**.

As far as A_n amplitudes are concerned, A_n/A_1 ratios (and φ_n phases) are characteristic of each $f(u)$ function.

In (7.30), the function $f(u)$ is a periodic function. For it to be a periodic wave, just make the substitution

$$u = t - \frac{x}{v}$$

➔ ***Fourier analysis***. *Every periodic wave $y(x - vt)$ can be decomposed into harmonic waves.*

Given the periodic wave $y(x - vt)$, we can always express it in the form

$$y(x - vt) = \sum_{n=1}^{\infty} A_n \sin(n\omega_1 t - nk_1 x + \varphi_n) \qquad v = \frac{\omega_n}{k_n} = \frac{n\omega_1}{nk_1} \tag{7.31}$$

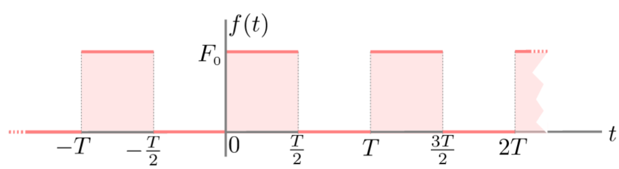


Fig. 7.23: Positive square wave

When we decompose a wave into harmonic waves, we perform a **Fourier analysis** of the wave. Combining this with the principle of superposition, we have:

➔ ***Fourier synthesis***. *Any periodic wave can be synthesised from the superposition of harmonic waves.*

When we construct a wave from harmonic waves, we perform a **Fourier synthesis** of the wave.

Example. Let $f(t)$ be the **positive square wave** (see Figure 7.23) defined as:

$$f(t) = \begin{cases} F_0\,, & \text{if } 0 < t < T/2 \\ 0\,, & \text{if } T/2 < t < T \end{cases}$$

It can be shown that the function can be decomposed into the following sine Fourier series:

$$f(t) = \frac{F_0}{2} + F_0\,\frac{2}{\pi}\left(\sin t + \frac{\sin 3t}{3} + \frac{\sin 5t}{5} + \dots\right) \tag{7.32}$$

In Figure 7.24, the first terms of the series (7.32), $n = 1, n = 2$ ... and the respective sums, $\sum_n$, have been plotted. In column (a) are only the first two terms, and in

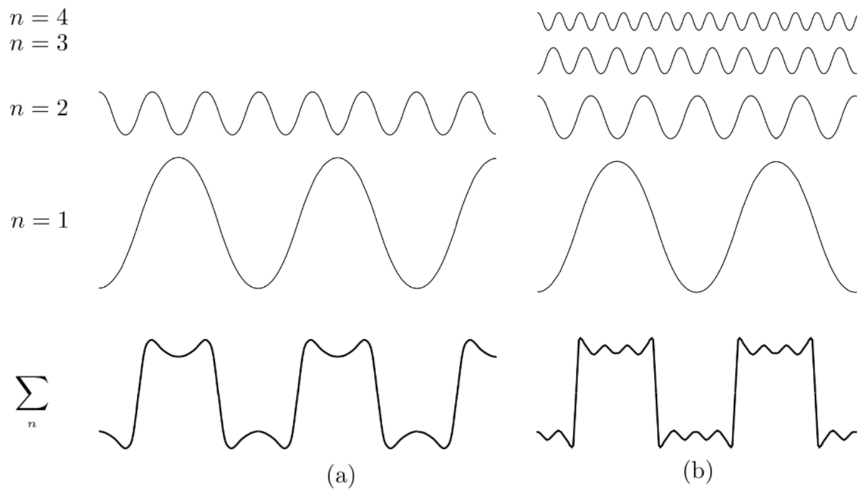


Fig. 7.24: First terms of the Fourier series corresponding to the positive square wave

column (b) are the first four. The sum of only four terms is a good approximation of the function (7.32). The graph in Figure 7.25, shows the relative amplitudes A_n/A_1, as a function of the frequency, ω_n, for $n = 1, 2, 3, \ldots$ The study of a periodic signal is called ***spectral analysis***.

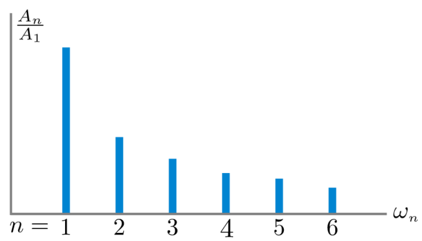


Fig. 7.25: Spectral analysis of a positive square wave

→ 8

8 Wave phenomena

Introduction

In this chapter, we will study phenomena related to waves, some of which are very closely linked to specific waves characteristics like the principle of superposition while others are related to aspects of energy like propagation and distribution through a medium. We will start by studying the latter.

8.1 The power and intensity of plane waves

We know that what propagates in a wave is a perturbation of one propertie in the medium. This always involves the propagation of energy.

The power of a harmonic wave on a string

We will now calculate the power carried by the harmonic wave $y(x,t) = A\sin(kx-\omega t)$ when propagating along a string of linear density μ and tension F. Note that, if there are no losses, this power must be the same as the power supplied by the focus of the waves.

As we have seen in Section 7.3, the transverse component y of the tension F exerted by the right-hand string section on the left-hand section is (see Figure 7.15)

$$F_y = F\,\sin\theta \approx F\,\tan\theta = F\,\frac{\partial y}{\partial x}$$

This force transmits, to the right, an instantaneous excitation power P_{exc} given by the product of F_y and the transverse velocity $\partial y/\partial t$ of each element of the string

$$P_{exc} = \frac{dW}{dt} = F_y\,\frac{\partial y}{\partial t} \approx F\,\frac{\partial y}{\partial x}\,\frac{\partial y}{\partial t} \tag{8.1}$$

If the wave is harmonic, (8.1) can be written as

$$P_{exc} = F\,k\omega A^2\,\cos^2(kx-\omega t)$$

Since $v = \sqrt{\frac{F}{\mu}}$ and $k = \omega/v$, the expression for P_{exc} can be written as

$$P_{exc} = \mu\omega^2 v\, A^2 \cos^2(kx - \omega t) \tag{8.2}$$

This power ranges from 0 to $\mu\omega^2 v\, A^2$. In practice, what is more interesting is the average power $\bar{P}_{exc}$, which we can calculate as the integral of P_{exc} over a period T divided by T, i.e,

$$\bar{P}_{exc} = \frac{1}{T}\int_t^{t+T} P_{exc}\, dt \tag{8.3}$$

Substituting (8.2), we find

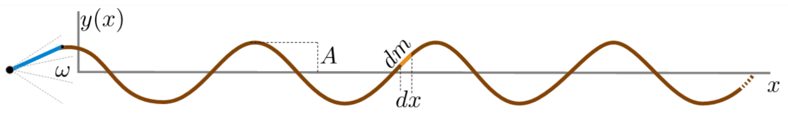
$$\bar{P}_{exc} = \frac{1}{2}\mu\omega^2 A^2\, v \tag{8.4}$$

The following problem finds the result (8.4) in a more direct albeit alternative way.

Problem 8.1.1. Calculate the energy per unit length of a string through which a harmonic wave propagates, and then the power required to be supplied to the string to maintain the wave

Solution

Solution to Problem 8.1.1



Consider a harmonic wave $y = A\sin(kx - \omega t)$ propagating at velocity v through a string of linear density μ. Each string differential can be understood as a particle of mass $dm = \mu\, dx$ subjected to a SHM of amplitude A and frequency ω (see Figure).

From the SHM study, we know that the element dm has the energy dE, given by

$$dE = \frac{1}{2}\left(\omega^2\, dm\right) A^2 = \frac{1}{2}\mu\omega^2 A^2\, dx$$

Therefore, the energy per unit length of the string is

$$\frac{dE}{dx} = \frac{1}{2}\mu\omega^2 A^2$$

In order to propagate a harmonic wave, energy must be released over time in a continuous and uninterrupted manner, for example, by attaching one end of the string to a vibrating plate controlled by a small motor. For a given point, the energy that during dt will pass at speed v will be that of the string element $dx = v\, dt$. The power, understood as the transported energy dE in a dt, $P = \frac{dE}{dt}$, will be

$$P = \frac{dE}{dt} = \frac{dE}{dx}\frac{dx}{dt} = \frac{1}{2}\mu\omega^2 A^2 v$$

which coincides with the average excitation power $\bar{P}_{exc}$ (8.4). ■

Intensity of a plane sound wave

What is the average power carried by a longitudinal harmonic wave propagating through a medium contained in a very long tube of straight section S? The medium can be a solid or a fluid. As in the previous example, we will rely on the SHM of a differential or slice of the tube medium, located at x and sized dx. If the medium has a density ρ, the mass of the differential is $dm = \rho\, S\, dx$ (see Figure 8.1).

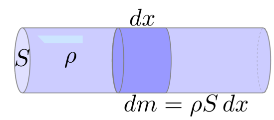


Fig. 8.1: Slice of the tube medium, at x and thickness dx

Since the mass dm located at x describes a SHM according to the harmonic wave $s(x,t) = A\sin(\omega t - kx)$, we can say the following.

- It has a kinetic energy dE_c:

$$dE_c = \frac{1}{2}\,dm\left(\frac{\partial s}{\partial t}\right)^2 = \frac{1}{2}\rho\, S\, dx\omega^2 A^2 \cos^2(\omega t - kx)$$

- It has an elastic potential energy dU. The mass is $dm = \rho\, S\, dx$. The associated potential energy is therefore

$$dU = \frac{1}{2}\,dm\,\omega^2\, s^2 = \frac{1}{2}\rho\, S\, dx\omega^2 A^2 \sin^2(\omega t - kx)$$

- The total energy is therefore

$$dE = dE_c + dU = \frac{1}{2}\,\rho\, dx\,\omega^2 A^2\, S$$

This energy is contained in the volume $dV = S\, dx = S\, v\, dt$. Therefore, $dm = \rho\, dV = \rho\, S\, v\, dt$. Thus, the energy passing through S at each dt is

$$dE = \frac{1}{2}\rho\omega^2 A^2\, S\, v\, dt$$

We define the **specific acoustic impedance** of the medium as

$$Z = \rho v \tag{8.5}$$

The above energy dE can be written as

$$dE = \frac{1}{2}Z\omega^2 A^2\, S\, dt$$

and the power of a plane harmonic wave across a transverse surface S as

$$P_s = \frac{1}{2}Z\omega^2 A^2\, S \tag{8.6}$$

We define the **intensity of a wave** as the power transmitted per unit of a cross-sectional area

$$I = \frac{dP_s}{dS} \tag{8.7}$$

The intensity associated with power (8.6) is

$$I = \frac{dP_s}{dS} = \frac{dE}{S\,dt} = \frac{1}{2}Z(\omega A)^2 \tag{8.8}$$

The SI unit for intensity is W/m². The unit of specific acoustic impedance (8.5) is kg/m²s = rayl (after Lord Rayleigh). Tables 7.1, 7.2 and 7.3 give the impedances of various media.

Intensity of plane sound waves as pressure waves

In (7.22) we saw that the acoustic pressure, p, is related to elongation s in the form

$$p = -B\,\frac{\partial s}{\partial x} \tag{8.9}$$

If the wave is harmonic, $s(x,t) = A\sin(\omega t - kx)$, then the acoustic pressure will be

$$p(x,t) = -B\,\frac{\partial s}{\partial x} = BAk\,\sin(\omega t - kx + \pi/2) \tag{8.10}$$

from which we find the following:

- The pressure wave, $p(x,t)$ is $\pi/2$ rad out of phase with respect to the displacement wave $s(x,t)$. When the elongation s is maximum, in absolute value, the acoustic pressure is zero and vice-versa.
- From (8.10) and (7.26), we find that the amplitude of the pressure wave, $\mathcal{P}$, is related to the amplitude of the displacement wave, A, in the form

$$\mathcal{P} = BAk = \rho v^2 A\,\frac{\omega}{v} = \rho\omega v\,A \tag{8.11}$$

The intensity of the sound waves, according to (8.8) and (8.5), can be expressed as a function of $\mathcal{P}$, taking into account (8.11):

$$I = \frac{1}{2}Z(\omega A)^2 = \frac{1}{2}\,\frac{\mathcal{P}^2}{Z} \tag{8.12}$$

Intensity level and decibels

In humans, the audible frequency range is 20 Hz to 20 kHz. Sounds below 250 Hz are qualified as **bass** or low tones. Above 5 kHz, they are qualified as **sharp** or high tones. Below 20 Hz, we have **infrasound**, and above 20 kHz is **ultrasound**.

The minimum audible intensity or threshold depends on the frequency. In humans, between 500 Hz and 5000 Hz is $I_0 = 10^{-12}$ W/m², and this increases when the frequency is far from this interval. Thus, the threshold intensity for 100 Hz or 18 kHz is of the order of 1000 times I_0.

In practice, to express the sound intensity received by a person, we use the **intensity level**, β, which is related to intensity I through the decimal logarithm:

$$\beta = 10 \log \frac{I}{I_0} \tag{8.13}$$

where I_0 is the aforementioned minimum intensity of 10^{-12} W/m^2 and β is expressed in decibels, dB.

The weakest β intensity level that can be captured is $\beta = 0$, which corresponds to $I = I_0$. The intensity level corresponding to the threshold of painful sound is 120 dB, corresponding to $I = 1$ W/m^2. Between 10 dB and 110 dB are all the usual intensities:

- In a quiet library, the sound can be between 30 and 40 dB.
- On a street with moderate traffic, it is between 50 and 70 dB.
- Near very noisy machinery, it is about 100 dB or more.

Figure 8.2 shows a sound-level meter, a device that measures intensity levels in everyday situations.

Fig. 8.2: Sound-level meter for measuring the sound intensity of everyday situations. Sound-level meters respond to the pressure of the sound wave

Problem 8.1.2. If the loudest and softest sounds we can sense are between 120 dB and 0 dB, what are the corresponding displacement wave and sound pressure wave amplitudes in the air?

Solution

Taking into account (8.12), the amplitudes A of the displacement and the sound pressure $\mathcal{P}$ as a function of the intensity of the sound are

$$A = \frac{1}{2\pi f}\sqrt{\frac{2I}{Z}} \quad , \quad \mathcal{P} = \sqrt{2ZI}$$

with $Z = 418$ rayl, as given in Table 7.3. Regarding the intensity, according to (8.13), we have

$$I = I_0\, 10^{\beta/10}\,, \qquad \text{with} \quad I_0 = 10^{-12} \ \text{W/m}^2$$

Using these expressions, we obtain the following table:

f	$\beta =$	0 dB	60 dB	120 dB
100 Hz	A (m)=	1.10×10^{-10}	1.10×10^{-7}	1.10×10^{-4}
5 kHz	A (m)=	2.20×10^{-12}	2.20×10^{-9}	2.20×10^{-6}
-	$\mathcal{P}$ (Pa)=	2.89×10^{-5}	2.89×10^{-2}	28.9

Solution table for Problem 8.1.2. Amplitudes for two frequencies and three levels of intensity

■

Absorption attenuation

When a wave propagates in an **absorbing medium**, there is a loss of energy called **attenuation by absorption**. Experimentally, we know that the loss of intensity of a plane wave, dI, as it passes through a thickness dx is proportional to the thickness and the incident intensity I. The proportionality factor is called the ***absorption coefficient*** of the medium, α:

$$dI = -\alpha\, I\, dx \tag{8.14}$$

where the sign indicates that the intensity decreases. Integrating (8.14), we have

$$\int_{I_0}^{I} \frac{dI}{I} = \int_0^x -\alpha\, dx \Rightarrow \boxed{I = I_0\, \mathrm{e}^{-\alpha\, x}} \tag{8.15}$$

The intensity of a plane wave decreases exponentially as it propagates through an absorbing medium.

8.2 The power and intensity of spherical waves

When the focus delivers a given power, it is uniformly distributed over the wave front. The wave intensity is the average power transmitted per unit of normal area to the direction of wave propagation. The intensity of the wave at the points on the sphere of radius r, centred at the focus, is

$$I = \frac{P}{4\pi r^2} \tag{8.16}$$

If the medium does not dissipate energy, the average power that passes through the surface of a sphere of radius r_1 must be the same as that which later passes through a sphere of radius r_2, with $r_2 > r_1$ (see Figure 8.3). If I_1 and I_2 are the respective intensities, we will have

$$P = 4\pi r_1^2\, I_1 = 4\pi r_2^2\, I_2$$

that is,

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2} \tag{8.17}$$

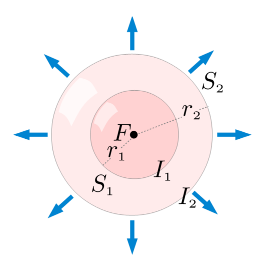


Fig. 8.3: A spherical wave

Therefore, even if the medium is not an absorbing medium, a spherical wave manifests a **geometric attenuation**.

➔ *The intensity of a spherical wave is inversely proportional to the square of the distance to the focus.*

Since the intensity is proportional to the square of the amplitude, the above result provides a physical argument for understanding that the amplitudes of spherical waves are inversely proportional to the distance r at the focus. If at $r = r_0$ the amplitude is s_0, then

$$s(r,t) = \frac{s_0\, r_0}{r} \sin(kr - \omega t) \tag{8.18}$$

where we have replaced the kx of the plane waves (7.8) with kr because the r coordinate now plays the same role as x in plane waves. The wave fronts are concentric spheres of radius r with the focus at $r = 0$.

In an absorbing medium, a spherical wave undergoes geometric and absorption attenuation. Taking into account (8.18) and (8.15), we have

$$I(r) = I_0 \frac{r_0^2}{r^2}\, \mathrm{e}^{-\alpha\, r} \qquad\qquad s(r,t) = A_0 \frac{r_0}{r}\, \mathrm{e}^{-\alpha r/2} \sin(kr - \omega t) \tag{8.19}$$

These expressions are valid for sound waves in air. Near the focus, the dominant attenuation is geometric. Far from the focus, the dominant attenuation is absorption

Problem 8.2.1. A loudspeaker emits sound at power $P = 0.5\,\mathrm{W}$ uniformly in all directions.

a) What is the intensity level of point A at a distance $10\,\mathrm{m}$ from the loudspeaker?

b) At what distance does the sound have an intensity level of $100\,\mathrm{dB}$?

c) How many loudspeakers with these same characteristics should work together in order to increase the intensity level at point A by $20\,\mathrm{dB}$?

Solution

a) Only expressions (8.13) and (8.16) need to be applied:

$$I = \frac{P}{4\pi r^2} = 3.98 \times 10^{-4}\ \mathrm{W/m^2}$$

$$\beta = 10 \log \frac{I}{I_0} = 10 \log \frac{3.98 \times 10^{-4}}{10^{-12}} = 86.0\ \mathrm{dB}$$

b) We will use (8.13) and (8.16) again, but now the unknown is r:

$$100 = 10 \log \frac{\left(\frac{0.5}{4\pi\, r^2}\right)}{10^{-12}}$$

Solving for r, it turns out that $r = 1.99\,\mathrm{m}$

c) If in **a)** the level was β, it will now be $\beta' = \beta + 20$. Since the intensity I of n (non-coherent) loudspeakers will be n times the intensity of a speaker, then

$$\beta + 20 = 10 \log \frac{nI}{I_0} = 10 \log n + 10 \log \frac{I}{I_0} = 10 \log n + \beta$$

from which we get

$$n = 10^{20/10} = 10^2 = 100$$

■

8.3 Transmission and reflection of a wave at a change in medium

In medium 1, a harmonic plane wave propagates in the direction of the x-axis. What happens when the wave hits $x = 0$, where there is a change in medium? For simplicity, let us consider the case where the boundary between media 1 and 2 is normal to the incident wave. The study of oblique incidence would give rise to the phenomenon of **refraction**.

We will analyse longitudinal waves in media 1 and 2, which we can characterise, respectively, by their acoustic impedances Z_1 and Z_2 and propagation velocities v_1 and v_2. The subscripts I, R and T will refer to incident, reflected and transmitted waves. The densities of the media are, according to (8.5), $\rho_1 = Z_1/v_1$ and $\rho_2 = Z_2/v_2$ and we will assume that the boundary is at $x = 0$ (see Figure 8.4).

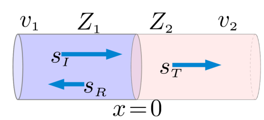


Fig. 8.4: Longitudinal waves encountering a change in medium

Incident, reflected and transmitted waves are

$$\begin{aligned} s_I(x - v_1 t) &= A_I \sin(\omega_I t - k_I x), \quad s_R(x + v_1 t) = A_R \sin(\omega_R t + k_R x + \varphi_R) \\ s_T(x - v_2 t) &= A_T \sin(\omega_T t - k_T x + \varphi_T) \end{aligned} \tag{8.20}$$

with the conditions

$$v_1 = \frac{\omega_I}{k_I} = \frac{\omega_R}{k_R} \qquad v_2 = \frac{\omega_T}{k_T} \tag{8.21}$$

Using the principle of superposition, we have

$$s_1(x,t) = s_I(x,t) + s_R(x,t) \qquad s_2(x,t) = s_T(x,t) \tag{8.22}$$

The following conditions must be satisfied:

a) Continuity of the medium: At any instant of time t, in the plane $x = 0$ between media, the following must be fulfilled:

$$s_1(0,t) = s_2(0,t) \tag{8.23}$$

which, taking into account (8.22) and (8.20), is

$$A_I \sin(\omega_I t) + A_R \sin(\omega_R t + \varphi_R) = A_T \sin(\omega_T t + \varphi_T) \tag{8.24}$$

b) Conservation of energy: If the media are not dissipative, the intensity of the reflected wave plus that of the transmitted wave must be equal to that of the incident wave: $I_I = I_R + I_T$. According to (8.8), we have

$$Z_1 \omega_R^2 A_R^2 + Z_2 \omega_T^2 A_T^2 = Z_1 \omega_I^2 A_I^2 \tag{8.25}$$

c) Conservation of momentum: Looking at Figure 8.4, we can consider the situation as a continuous series of shocks occurring from left to right. At $x = 0$, and for all t, in a dt the mass $dm_1 = \rho_1 S v_1 dt$ (S is the tube section) moving at an oscillation velocity of $\frac{\partial s_I}{\partial t}(0,t)$ collides with a mass $dm_2 = \rho_2 S v_2 dt$ at rest. As a consequence of the collision, dm_1 and dm_2 have velocities $\frac{\partial s_R}{\partial t}(0,t)$ and $\frac{\partial s_T}{\partial t}(0,t)$, respectively. The conservation of momentum can be written as follows:

$$\frac{Z_1^2}{\rho_1}\frac{\partial s_1}{\partial x}(0,t) = \frac{Z_2^2}{\rho_2}\frac{\partial s_2}{\partial x}(0,t) \tag{8.26}$$

which, if necessary, we can make explicit with the expression (8.20).

With these three conditions, the following can be proved:

1) The frequencies ω_R and ω_T are equal to the incident frequency, ω_I:

$$\omega_I = \omega_R = \omega_T \ \equiv \omega \tag{8.27}$$

2) The phase shifts φ_R and φ_T are 0 or π. Since $\varphi = \pi$ is equivalent to $\varphi = 0$ with a negative amplitude, we take $\varphi_R = \varphi_T = 0$ and allow for the possibility that $A_{TR} \lesseqgtr 0$.

Considering (8.24) for the particular case $\omega t = \pi/2$, and (8.25) we find the equalities

$$A_I + A_R = A_T \quad , \quad Z_1 A_R^2 + Z_2 A_T^2 = Z_1 A_I^2 \tag{8.28}$$

from which we obtain

$$\frac{A_R}{A_I} = a_R \qquad \text{with} \qquad a_R = \frac{Z_1 - Z_2}{Z_1 + Z_2} \tag{8.29}$$

$$\frac{A_T}{A_I} = a_T \qquad \text{with} \qquad a_T = \frac{2Z_1}{Z_1 + Z_2} \tag{8.30}$$

where a_R and a_T are the **amplitude reflection and transmission coefficients**.

The incident, reflected and transmitted wave intensities are

$$I_I = \frac{1}{2} Z_1 \omega^2 A_I^2 \qquad I_R = \frac{1}{2} Z_1 \omega^2 A_R^2 \qquad I_T = \frac{1}{2} Z_2 \omega^2 A_T^2$$

Taking into account (8.29) and (8.30), we find

$$\frac{I_R}{I_I} = p_R \qquad \text{with} \qquad p_R = a_R^2 = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2 \tag{8.31}$$

$$\frac{I_T}{I_I} = p_T \qquad \text{with} \qquad p_T = \frac{Z_2}{Z_1} a_T^2 = \frac{4 Z_1 Z_2}{(Z_1 + Z_2)^2} \tag{8.32}$$

where p_R and p_T are the **power reflection and transmission coefficients**.

With these relations, we can build Table 8.1 for cases according to the relative value of the impedances Z_1 and Z_2.

We can also state the following:

a)	If $Z_1 \sim Z_2$:	$a_R \approx 0,$	$a_T \approx 1,$	$p_R \approx 0,$	$p_T \approx 1$
b)	If $Z_1 \ll Z_2$:	$a_R \approx -1,$	$a_T \approx 0,$	$p_R \approx 1,$	$p_T \approx 0$
c)	If $Z_1 < Z_2$:	$a_R < 0,$	$a_T > 0,$		
d)	If $Z_1 > Z_2$:	$a_R > 0,$	$a_T > 0,$		

Table 8.1: Reflection and transmission coefficients for different impedance ratios

a) If the impedances are similar, practically all the intensity passes into the transmitted wave.

b) If the impedances are very different, practically all the intensity is reflected (see an example in Problem 8.3.1).

c) If $Z_1 < Z_2$, then $A_R < 0$, i.e., the reflected wave undergoes a phase shift of π.

The reflection and transmission coefficients for harmonic waves have been found to be frequency independent. Consequently, the coefficients obtained can also be applied to non-harmonic waves.

Problem 8.3.1. A plane sound wave propagating through the air is normally incident on the water in a pool. What percentage of intensity passes into the water? How many decibels does the intensity level decrease as it passes into the water?

Solution

According to Tables 7.3 and 7.2, the impedance of air is 418 rayl, and that of water is 1.45×10^6 rayl. Using (8.32), we find that the power passing into the water is

$$p_T = 0.00115 = 0.115\%$$

Thus 99.88% of the intensity is reflected.

According to (8.13), the loss of intensity level when passing through water is

$$\Delta\beta = 10 \log \frac{I_{\text{water}}}{I_0} - 10 \log \frac{I_{\text{air}}}{I_0} = 10 \log \frac{I_{\text{water}}}{I_{\text{air}}} = 10 \log p_T = -29.4 \text{ dB} \quad \blacksquare$$

8.4 Interference and beats

Superposition of two harmonic waves

When two harmonic waves of the same frequency ω meet at a point, P, the elongation at this point oscillates harmonically (see Figure 8.5).

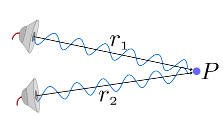


Fig. 8.5: Two harmonic plane waves meet at a point P

If point P is at a distance r_1 from focus F_1 and at r_2 from focus F_2, the elongations at point P can be written as

$$s_1(r_1, t) = A_1 \sin(\omega t - kr_1 + \varphi_1) \qquad s_2(r_2, t) = A_2 \sin(\omega t - kr_2 + \varphi_2) \quad (8.33)$$

The resulting elongation at P is the superposition of s_1 and s_2

$$s(P, t) = s_1(r_1, t) + s_2(r_2, t)$$

Using the notation $\phi_1 = -kr_1 + \varphi_1$ and $\phi_2 = -kr_2 + \varphi_2$, we have

$$\begin{aligned} s(P,t) &= A_1 \sin(\omega t + \phi_1) + A_2 \sin(\omega t + \phi_2) = \\ &= \sin\omega t[A_1 \cos\phi_1 + A_2 \cos\phi_2] + \cos\omega t[A_1 \sin\phi_1 + A_2 \sin\phi_2] \end{aligned} \tag{8.34}$$

We want to write this sum in the form

$$s(P,t) = A_R \sin(\omega t + \phi) \tag{8.35}$$

where we need A_R and ϕ. To do so, we develop (8.35)

$$s(P,t) = \sin\omega t\, A_R \cos\phi + \cos\omega t\, A_R \sin\phi$$

and use (8.34) to identify the following:

$$\tan\phi = \frac{A_1 \sin\phi_1 + A_2 \sin\phi_2}{A_1 \cos\phi_1 + A_2 \cos\phi_2} \tag{8.36}$$

$$A_R^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_1 - \phi_2) \tag{8.37}$$

In the simplest case, where $A_1 = A_2 = A$, then we have

$$A_R = 2A\cos\left(\frac{\phi_1 - \phi_2}{2}\right) \qquad \phi = \frac{\phi_1 + \phi_2}{2} \tag{8.38}$$

Interferences

Suppose the two amplitudes are equal. $s(P,t)$ is

$$s(P,t) = A_R \sin(\omega t + \phi) = 2A\cos\left(\frac{\phi_1 - \phi_2}{2}\right)\sin\left(\omega t + \frac{\phi_1 + \phi_2}{2}\right)$$

Substituting ϕ_1 and ϕ_2 and making $\Delta r = r_2 - r_1$, $\Delta\varphi = \varphi_2 - \varphi_1$, we obtain[1]

$$s(P,t) = 2A\cos\left(\frac{k\Delta r}{2} - \frac{\Delta\varphi}{2}\right)\sin\left(\omega t + \frac{\varphi_1 + \varphi_2}{2}\right)$$

$s(P,t)$ is a SHM of amplitude

$$A_R = 2A\cos\left(\frac{k\Delta r}{2} - \frac{\Delta\varphi}{2}\right) \tag{8.39}$$

A_R depends on the difference in paths travelled, Δr, and the initial phase difference between the two waves, $\Delta\varphi$. We highlight the following:

- **Constructive interference**: The amplitude A_R is maximum, $A_{R_{\max}} = 2A$, if

$$\pi\frac{\Delta r}{\lambda} - \frac{\Delta\varphi}{2} = 2n\,\frac{\pi}{2}\,, \qquad n = 0,\ \pm1,\ \pm2,\ \ldots \tag{8.40}$$

[1] Let's remember that $\sin\alpha + \sin\beta = 2\cos\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}$

- **Destructive interference**: The amplitude A_R is minimum, $A_{R_{\min}} = 0$, if

$$\pi \frac{\Delta r}{\lambda} - \frac{\Delta \varphi}{2} = (2n+1)\,\frac{\pi}{2}\,, \qquad n = 0,\ \pm 1,\ \pm 2,\ \ldots \tag{8.41}$$

We also highlight the following two very common cases:

- When both waves are *in phase*: $\Delta\varphi = 0 + 2n\,\pi,\ n = \pm 1,\ \pm 2,\ \ldots$,

$$\text{Constructive interf.:} \qquad \Delta r = \quad 2n\,\frac{\lambda}{2} \tag{8.42}$$

$$\text{Destructive interf.:} \qquad \Delta r = \quad (2n+1)\,\frac{\lambda}{2} \tag{8.43}$$

- When both waves are *in opposite phase*: $\Delta\varphi = \pi \pm 2n\pi$:

$$\text{Constructive interf.} \qquad \Delta r = \quad (2n+1)\,\frac{\lambda}{2} \tag{8.44}$$

$$\text{Destructive interf.:} \qquad \Delta r = \quad 2n\,\frac{\lambda}{2} \tag{8.45}$$

If the amplitudes are not equal, the interference conditions are the same. Neither the minimum amplitude, $A_{R_{\min}}$, will be zero nor the maximum amplitude, $A_{R_{\max}}$, will be $2A$. We will obtain

$$A_{R_{\min}} = \sqrt{A_1^2 + A_2^2 - 2A_1A_2} = |A_1 - A_2| \tag{8.46}$$

$$A_{R_{\max}} = \sqrt{A_1^2 + A_2^2 + 2A_1A_2} = A_1 + A_2 \tag{8.47}$$

Interferences and coherence

In all that we have said about interference, one fundamental detail stands out: The initial phase shift $\Delta\varphi = \varphi_2 - \varphi_1$ must be constant over time, i.e., both foci must be **coherent sources**. Coherent sources are foci that emit waves whose phase differences are independent of time.

Problem 8.4.1. Two people are singing with the same power. Although they emit the same frequency, the waves emitted will not be coherent.

a) If at a point P, equidistant from the two people, the intensity of one singing is I, how much is the intensity at point P when both sing and how many decibels does the intensity level increase?

The waves are now emitted by two coherent loudspeakers of the same frequency and generate constructive interference at P.

b) What is the total intensity at P? What is the intensity level?

Solution

a) As there is no coherence condition, there is no interference and the intensities are proportional to the energy. The total intensity is the sum of the two intensities, i.e., $I_{\text{total}} = I + I = 2I$.

The intensity level when going from one voice to two (that is, from I to $2I$), increases by: $\Delta\beta_{nc} = 10 \ln 2 = 3.0\,\text{dB}$. The subscript $_{nc}$ stands for non-coherent.

b) In a harmonic wave, the intensity is proportional to the square of the amplitude A: $I = \alpha A^2$. If there is constructive interference at P, the amplitude at P will be $2A$ and the corresponding intensity will be

$$I_{\text{total}} = \alpha(2A)^2 = 4\,\alpha A^2 = 4I$$

And the intensity level is increased by $\Delta\beta_c = 10 \ln 2^2 = 20 \ln 2 = 2\Delta\beta_{nc} = 6.0\,\text{dB}$.

■

Problem 8.4.2. Two loudspeakers, A and B, coherent and in phase, emit sound waves of the same frequency arriving with the same intensity at a listener O located 4 m from A and 3 m from B. Which frequencies f_n will the listener not be able to hear?

Fact: The temperature is 15°C; therefore, the speed of sound is 340 m/s.

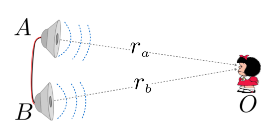


Figure for Problem 8.4.2

Solution

Since the waves emitted from the loudspeakers are coherent they will experience interference. The frequencies f that the listener will hear either with difficulty or not at all will be those that cause destructive interference. Therefore, since the loudspeakers are in phase, we must apply the equality (8.43): $\Delta r = (2n+1)\lambda/2$ with $n = 0, \pm 1, \pm 2, \ldots$ Since $\lambda = v/f$ and $\Delta r = r_b - r_a = 1\,\text{m}$, the destructive interference condition is

$$\Delta r = (2n+1)\frac{\lambda}{2} = (2n+1)\frac{v}{2f}$$

that is,

$$f_{\text{dest}} = (2n+1)\frac{v}{2\,\Delta r}$$

With the data, $\Delta r = 1\,\text{m}$, $v = 340\,\text{m/s}$, and $(2n+1) = 1, 3, 5, \ldots$, we obtain

$$f_{\text{dest}} = 170\,\text{Hz},\ 510\,\text{Hz},\ 850\,\text{Hz},\ 1190\,\text{Hz},\ldots\ \text{etc.}$$

■

Beats

Beats are produced when we superimpose two harmonic waves of slightly different frequencies at a point P. Suppose that, at P, they have the same amplitude (see Figure 8.5 on page 198)

$$s_1(r_1,t) = A\ \sin(\omega_1 t - k_1 r_1 + \varphi_1) \qquad s_2(r_2,t) = A\ \sin(\omega_2 t - k_2 r_2 + \varphi_1) \tag{8.48}$$

with $\omega_2 \gtrsim \omega_1$: $\omega_2 = \omega_1 + \Delta\omega$ with $\Delta\omega \ll \omega = \frac{\omega_1+\omega_2}{2}$. Adding s_1 and s_2, with $\phi_1 = -k_1 r_1 + \varphi_1$ and $\phi_2 = -k_2 r_2 + \varphi_2$, we obtain the elongation of point P, $s(P,t)$,

$$s(P,t) = 2A\cos\left(\frac{t\,\Delta\omega}{2} + \frac{\phi_1 - \phi_2}{2}\right)\sin\left(\omega t + \frac{\phi_1+\phi_2}{2}\right) \tag{8.49}$$

which is similar to that of a SHM of frequency ω, practically the same as that of the original waves, but with an amplitude A_R

$$A_R = 2A\cos\left(\frac{t\,\Delta\omega}{2} + \phi\right) \tag{8.50}$$

A_R is harmonically time-dependent, with a frequency $\Delta\omega/2$. ϕ is a phase containing the dependence on r_1 and r_2, and it is constant for fixed point P.

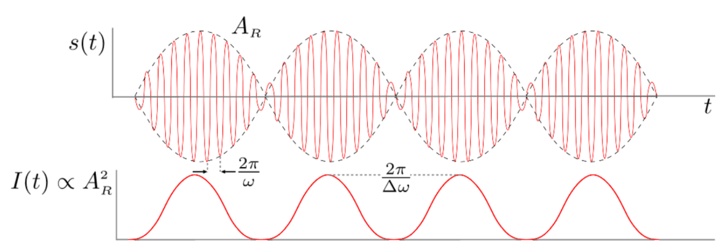


Fig. 8.6: Beats. Graphs for elongation and intensity

The intensity $I \propto A_R^2$ has a period that is half that of the amplitude A_R (see the graph in Figure 8.6). Thus, continuing with the example of sound waves, the receiver located at P perceives the intensity of the beats with a frequency twice that of (8.50):

$$\omega_{\text{beat}} = |\omega_2 - \omega_1| = \Delta\omega \tag{8.51}$$

A very simple way to produce beats is shown in Figure 8.7 where we have the simultaneous sounding of two tuning forks of slightly different frequencies, e.g., $f = 440\,\text{Hz}$ and $f' = 437\,\text{Hz}$. The beats will cause the intensity to vary with a frequency $f_{\text{beat}} = \Delta f = 440 - 437 = 3\,\text{Hz}$.

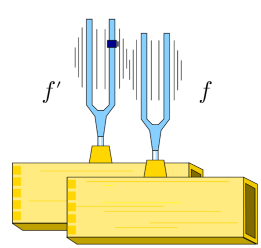


Fig. 8.7: Producing of beats with two very similar tuning forks

8.5 Standing waves

In Section 8.3 we have seen that part of a wave is reflected when it arrives at a change in medium. If the impedances are very different, practically the whole wave is reflected. This is what happens when a wave reaches the end of the medium and the next medium is very different. Since in reality all media are bounded, the propagation of the initial wave will have to be superimposed over the propagation of the reflected wave at the end of the medium. The latter may have a phase shift of π rad.

We study the superposition of two harmonic plane waves of the same amplitude and frequency but propagating in opposite direction. One can be the initial wave and the other the reflected wave. We have

$$y_1(x,t) = A\sin(\omega t - kx + \varphi_1) \qquad y_2(x,t) = A\sin(\omega t + kx + \varphi_2) \tag{8.52}$$

where φ_1 and φ_2 are constants. Applying (8.35, 8.38), we obtain that $y = y_1 + y_2$ is

$$y(x,t) = 2A\,\cos(kx+\phi)\sin(\omega t + \phi') \tag{8.53}$$

where $\phi = (\varphi_1 - \varphi_2)/2$, $\phi' = (\varphi_1 + \varphi_2)/2$. We will see that ϕ and ϕ' are irrelevant.

The superposition of two waves is always a wave. Therefore, (8.53) is a wave. But we also see that this wave does not propagate: every point x performs a SHM of frequency ω and amplitude A_R which depends on x in the form

$$A_R(x) = 2A\,\cos(kx+\phi) \tag{8.54}$$

but which does not depend on t. There are points that never oscillate, $A_R = 0$; they are the **nodes**. Others always oscillate with the maximum amplitude, $2A$; they are the **bellies or antinodes**.

Since $k = 2\pi/\lambda$, the positions of the antinodes and nodes are, according to (8.54),

$$\begin{cases} \text{Antinodes:} \quad \cos(kx+\phi) = \pm 1, \quad \text{from which} \quad \frac{2\pi}{\lambda}x_n + \phi = n\,\pi \\ \text{Nodes:} \quad \cos(kx+\phi) = 0, \quad \text{from which} \quad \frac{2\pi}{\lambda}x_n + \phi = (2n+1)\frac{\pi}{2} \end{cases}$$

where $n = 0, \pm 1, \pm 2, \ldots$. The distance between two nodes, or two consecutive antinodes, n and $n+1$, is

$$x_{n+1} - x_n = \frac{\lambda}{2} \tag{8.55}$$

Figure 8.8 shows a medium experiencing standing waves. In it we see four nodes (N) and three antinodes (V). The figure shows the graph of the perturbation $y(x, t_i)$ at six different time instants $i = 1.2, \ldots$.

Standing waves in finite string

Consider a string of length L, subjected to a tension F and with both ends fixed, meaning the ends do not move. We vibrate a point on the string and produce a

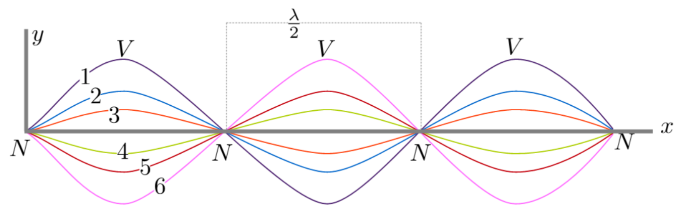


Fig. 8.8: Standing waves in a medium. The different instants have been highlighted with different colours

harmonic wave. This is the initial wave, which will reach one of the ends and be reflected, thus superimposing the initial and reflected waves along the whole length of the string. This superposition gives rise to standing waves, but we will also have to see what happens to the reflected wave when it reaches the opposite end of the string, because it will give rise to a new reflection that will be superimposed over the previous ones. This will go on indefinitely. In general, a string of length L has no standing waves. The standing wave solution will occur only when two of the N nodes coincide with the fixed ends, which happens only for certain frequencies. Figure 8.9 shows three possible cases of a string with fixed ends subjected to standing waves. These are, namely, when the string's length L is an integer multiple of half a wavelength: the distance between node and node or the distance between antinode and antinode. That is to say,

$$L = n\,\frac{\lambda}{2}\,, \qquad \text{on} \quad n = 1,\ 2,\ 3,\ \ldots \tag{8.56}$$

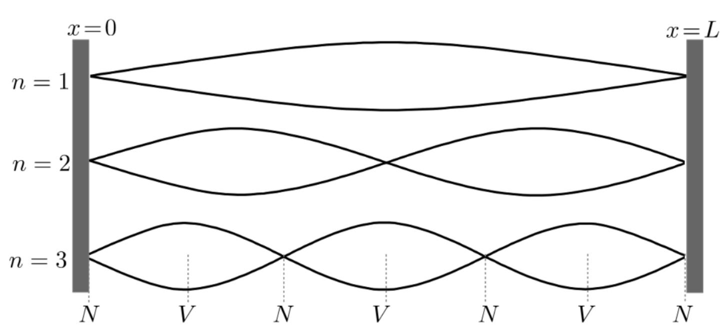


Fig. 8.9: Three possible cases of standing waves on a string with fixed ends

Only λ wavelengths satisfying (8.56) are possible. If the velocity of the waves on the string is $v = \sqrt{F/\mu}$, with $\lambda = v/f$, the only possible frequencies f that will produce standing waves are

$$f_n = n\,\frac{v}{2L} \tag{8.57}$$

where $n = 1,\ 2,\ 3,\ \ldots$.

➔ $f_1 = \frac{v}{2L}$ is the lowest frequency or **fundamental frequency**. $n = 2$, $n = 3$... are the **second harmonic**, the **third harmonic**, and what are generally called higher harmonics. The set of frequencies f_n are the **eigenfrequencies**, and the corresponding oscillation modes are the **normal modes of oscillation**. The eigenfrequencies are often also referred to as **resonance frequencies**.

In Figure 8.9, we can see a schematic of the string in the fundamental, second and third modes.

Problem 8.5.1. The figure shows a string, of density $0.65\,\mathrm{g/m}$, with one end fixed to a wall and, after the other end passes through a pulley, it has a weight mg attached to it. Both ends can be considered fixed. The length of the string up to the pulley is $L = 31.6\,\mathrm{cm}$. Next to the string we have placed a loudspeaker, A, which emits a sound that we vary between $500\,\mathrm{Hz}$ and $1500\,\mathrm{Hz}$. We observe that the string resonates only at frequencies of $880\,\mathrm{Hz}$ and $1320\,\mathrm{Hz}$. Find the tension to which the string is subjected and to which harmonics these frequencies correspond.

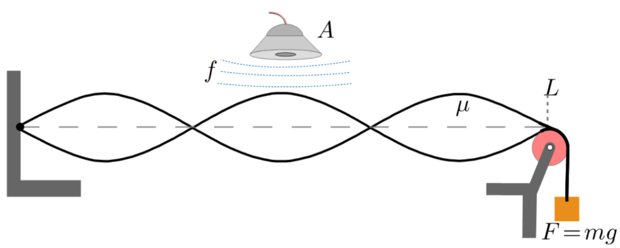


Figure for Problem 8.5.1

Solution

If the frequency of loudspeaker A is continuously varied and resonance occurs only at 880 Hz and 1320 Hz, then these frequencies correspond to two consecutive values of the possible standing wave frequencies, namely one corresponding to mode n and another to $n+1$ although we generally do not know the natural value corresponding to n. Now, through (8.57), we have

$$f_{n+1} - f_n = f_1 = \frac{v}{2L} = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$

that is to say,

$$1320 - 880 = \frac{1}{2 \cdot 0.316}\sqrt{\frac{mg}{0.65 \times 10^{-3}}}$$

from which we have that the tension of the string is: $F = mg = 50.26\,\mathrm{N}$.

Since $f_1 = 440\,\mathrm{Hz}$, the harmonic number of the resonant frequencies found is $n = \frac{880\ \mathrm{Hz}}{440\ \mathrm{Hz}} = 2$ and $n + 1 = 3$ ■

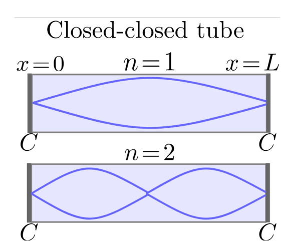


Fig. 8.10: Tube closed at both ends

Standing waves in finite tubes

Here, we study the formation of longitudinal standing waves in a gaseous medium contained in a tube. If the tube is **closed at both ends**, the air within that is in contact with the tube walls cannot vibrate. The behaviour of the displacement waves in this tube is the same as that of waves in a string with fixed ends. If L is the length of the tube and $v = \sqrt{\frac{\gamma RT}{M}}$ the propagation speed, the frequencies f_n producing standing waves are those given by (8.57). Figure 8.10 shows a tube of length L, closed at both ends and vibrating according to the first and second harmonics of the displacement wave.

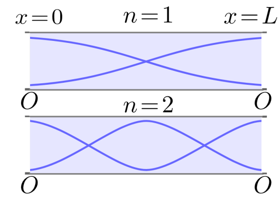


Fig. 8.11: Tubes open at both ends

If the ends are open, we have what is known as a **mouth effect**, in which a displacement antinode forms at a small distance from these ends, proportional to the diameter of the tube. For the sake of simplicity, we will not take into account this effect. Figure 8.11 shows the first two harmonics for this case, where we can see that the positions of the nodes and antinodes are transposed with respect to the case of the tube closed at both ends. The possible frequencies for standing waves (8.57) are the same as those corresponding to the same tube with both ends closed.

➔ *In closed-closed and open-open tubes, all harmonics are possible.*

If one end is open and the other closed, there will be a displacement node at the closed end and an antinode at the open end. In Figure 8.12, the schematics are shown for the first three major wavelength cases. If $n = 1, 2, 3, ...$ is the harmonic sequence (8.57), the possible n_s harmonics in the closed-open tube must fulfil

$$L = (2n-1)\frac{\lambda}{4} = n_s\frac{\lambda}{4} \qquad \text{on} \qquad n_s = 1, 3, 5, ...$$

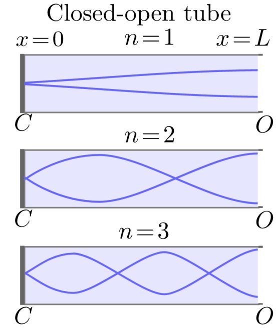


Fig. 8.12: Tube with one end open and the other closed

And, since $v = \lambda f$, then the only frequencies f that give rise to standing waves in a closed-open tube are

$$f_{n_s} = n_s\frac{v}{4L}\ ,\ n_s = 1, 3, 5... \tag{8.58}$$

➔ The lowest frequency or **fundamental frequency**, $n_s = 1$, is $f_1 = \frac{v}{4L}$. The frequencies of the upper harmonics are obtained from (8.58).

➔ *In a closed-open tube, only odd harmonics are possible.*

➔ Since the pressure wave is $\pi/2$ rad out of phase with respect to the displacement wave, the nodes and antinodes of the pressure wave are interchanged with respect to those of the displacement wave.

Problem 8.5.2. The length of the soundboard of a tuning fork, closed at one end and with a frequency of 440 Hz is 17.0 cm. Is this length appropriate?


Figure for Problem 8.5.2

Solution

Taking the speed of sound at 343 m/s when the air is 20°C, and applying (8.58), for $n_s = 1, 3, 5...$, we have

$$f_{n_s} = 440\ \text{Hz} = n_s\frac{v}{4L}$$

from which

$$L = n_s \frac{343}{4 \cdot 440} = n_s 0.195 \text{ m}$$

For $n_s = 1$, we get 19.5 cm $\neq$ 17.0 cm. The length of the box does not seem to be adequate, although it is if we were to take into account the mouth effect. ■

Resonance

In much the same way as in the one-degree-of-freedom oscillating systems studied in Chapter 6 (point or 0-dimensional objects), one-, two- and three-dimensional objects can also experience resonance phenomena. In this chapter we have studied one-dimensional objects (strings, rods, tubes, etc.), but what we have discussed can be extended to other dimensions.

Any object is a limited medium susceptible to vibration through the excitation of waves arriving from adjacent media. Its geometrical characteristics and materials will grant this object with its own frequencies. We say that the object **enters into resonance** when the external exciter vibrates at one of the object's own frequencies and causes standing waves of this frequency.

A good example of this can be found in the behaviour of musical instruments.

➔ The fundamental frequency f_1 of a musical instrument corresponds to the **tone** or **musical note**. The ratio of the amplitudes of the higher harmonics to the fundamental, A_2/A_1, A_3/A_1... determines the **timbre** of the instrument. A clarinet C and a flute C have the same pitch but different timbre.

Figure 8.13 shows the spectral analysis of the C_4 of a piano and the corresponding waveform. Notice that the dominant tone is the fundamental harmonic, 261 Hz, and that the amplitudes of the next two harmonics decrease rapidly. There are instruments that are richer in harmonics, i.e., the A_n/A_1 sequence decreases more slowly.

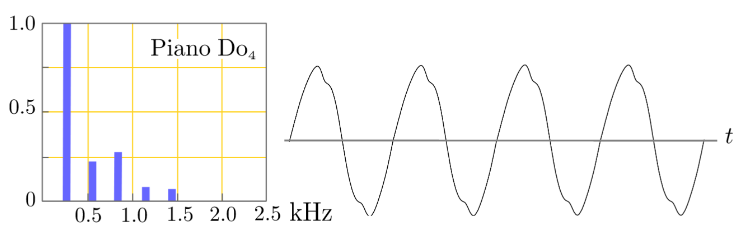


Fig. 8.13: A mplitudes relationship for piano's central C or C_4 and the wave resulting from the superposition

In many cases, resonances involve such large amplitudes and/or accelerations that they can be destructive and should be avoided. In other cases, resonances can be very useful. Thus, a system with different resonance frequencies can act as a frequency **filter**.

An example of a filter is shown in Figure 8.14. A very long string is attached at its right end to an elastic bar that is clamped in the middle. At the opposite symmetrical end of the bar, a second very long string is attached. A progressive wave y_I of frequency ω_I arrives at the left-hand end of the string. This incident wave excites the bar. The bar, with the fixed centre point (node) and free ends

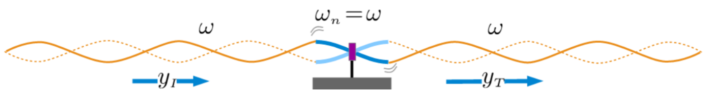


Fig. 8.14: A very long string attached to an elastic bar

(antinodes), can vibrate transversely with a given set of frequencies[2] f_n. If the frequency of the incident wave coincides with one of the bar's vibration modes, transverse standing waves will be produced in the bar, thus giving rise to a wave y_T being transmitted to the right-hand string. In this case, we would say that there is resonance between the incident wave and the bar. Otherwise, the bar would not vibrate (at least not noticeably) and there would be no transmitted wave.

[2] For longitudinal vibrations, a bar under the described conditions vibrates at the frequencies $f_n = nf_1$, with $f_1 = \frac{v}{2L}$ and $n = 1, 2, 3, \ldots$, i.e., the same frequencies at which a tube vibrates with both ends closed or open. Transverse vibrations are more complex than longitudinal vibrations and the relation $f_n \neq nf_1$ is not valid, but some f_n frequencies exist

If the frequency of the incident wave coincides with one of the bar's own frequencies, $\omega = \omega_n$, the waves in the strings will be transmitted, i.e.,

$$y_I(x,t) = A\cos(\omega t - kx)$$
$$y_{\text{bar}} = A\cos k_n x \cos \omega_n t$$
$$y_T(x,t) = A\cos(\omega t - k\,[x - L] + \pi)$$

If it does not match, there will be no transmitted wave. If a superposition of many frequencies reaches the left-hand string, the only waves that the bar will transmit to the right-hand string are those whose frequencies coincide with one of the bar's frequencies. Changing the bar will change the frequencies that will be transmitted and, thus, the bar acts as a filter that favours the passage of some frequencies while preventing all others.

Problem 8.5.3. In the figure are two rooms separated by a wall, P, which insulates the rooms acoustically, except for a narrow tube of length L that is open at both ends and connecting them. One of the rooms has near the tube, a generator, D, which emits sound at frequency f and intensity I_a.

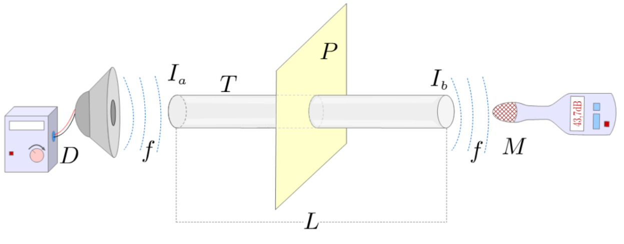


Figure for Problem 8.5.3

In the other room, a sound-level meter M measures the I_b of the sound coming through the tube. The emission and the measurement of I_b is then repeated for various frequencies f. Assuming that all the intensities I_a emitted are equal, what are the measured intensities I_b as a function of frequency f?

Solution

Each received intensity I_b depends strongly on the frequency f of the sound. If this is the fundamental frequency of the tube, $f_1 = v/2L$, or one of its multiples, $f_n = n\, f_1$, there will be resonance and the intensity I_b will be the maximum. But, as the frequency f moves away from one of these tube eigenfrequencies, the intensity I_b of the sound decreases to a minimum. If the radius of the tube is relatively large, the maximum I_b will be just below I_a and the minimum will be close to zero. The tube acts as an acoustic filter. ■

8.6 Doppler effect and shock waves

Doppler effect

Fig. 8.15: Christian Andreas Doppler (1803-1853) was an Austrian mathematician and physicist

The **Doppler effect** is the variation in the frequency of the waves received by an observer due to the relative movement between him and the source. When a train is approaching us, its whistle is more high-pitched than when it is stationary, and it is more low-pitched when it is moving away.

In Figure 8.16, the observer O measures the sequence f_O of incoming waves, emitted by source F with a frequency f_F.

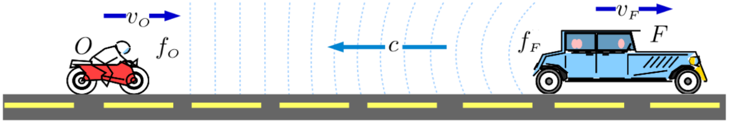


Fig. 8.16: Doppler effect

Let c be the wave velocity and let v_F and v_O be the constant velocities of F and O (positive in the direction of O towards F).

Suppose that the source F emits a wavefront at a certain instant t_1; then, after a period $T_F = 1/f_F$ and at the instant $t_2 = t_1 + T_F$, it emits the next one, and so on in a similar manner. If x_{F0} is the distance from the observer to the source at instant t_1, these wave fronts will reach the observer-receiver at instants t_1', t_2', etc., given by the expressions

$$v_O\,(t_1'-t_1) = x_{F0}-c\,(t_1'-t_1) \qquad v_O\,(t_2'-t_1) = x_{F0}+v_F T_F-c\,(t_2'-t_1-T_F)$$

From this we find that the period T_O of the wave reaching the observer is

$$T_O = t_2' - t_1' = \frac{c+v_F}{c+v_O}\,T_F$$

or, in terms of frequencies,

$$\frac{f_O}{c+v_O} = \frac{f_F}{c+v_F} \tag{8.59}$$

It should be noted here that the velocities v_O and v_F are relative to the medium. If we are dealing with sound waves and the medium is air, the expression (8.59) must use the velocities relative to the air, while also taking into account the wind.

The Doppler effect has many practical applications, such as radars to detect speeding in traffic, sonars used in maritime navigation and ultrasound scans, just to name a few. A large part of astrophysics knowledge is based on the Doppler effect experienced by the electromagnetic waves emitted by stars and galaxies moving relative to the Earth.

Problem 8.6.1. Two students, A and B, carry equal tuning forks vibrating at $f_d = 440\,\text{Hz}$. If A is standing still and B is moving away from her at $v_B = 6\,\text{m/s}$, how many beats per second will each one feel as a result of the Doppler effect? Assume that there is no wind and that the speed of sound is $c = 340\,\text{m/s}$.

Solution

Applying the above expression (8.59) to the Doppler effect in both cases, it follows that the tunning fork frequencies felt by A and B, respectively f'_A and f'_B, are

$$f'_A = \frac{c}{c+v_B} f_d \qquad f'_B = \frac{c-v_B}{c} f_d$$

Note that if v_B is much smaller than c, the two frequencies are very similar. The frequencies of the beats are

$$f_A^{\text{beat}} = f_d - f'_A = \frac{v_B}{c+v_B} f_d = 7.63\,\text{Hz}\ ;\ f_B^{\text{beat}} = f_d - f'_f = \frac{v_B}{c} f_d = 7.76\,\text{Hz} \quad \blacksquare$$

Shock waves

Shock waves are produced when the emitting source moves faster than the waves it sends out. Examples of shock waves are those produced by aircraft travelling faster than the speed of sound or the two branches of a wake left by a ship moving through the water.

In Figure 8.17, a source F of spherical waves moves in the direction of the x-axis at a constant velocity v_F, which is greater than the velocity c of the emitted waves. At instant $t = 0$, the source F is at $x_0 = 0$ and emits a wave front, which at instant $t_N = N\Delta t$ (in Figure 8.17, we use $N = 5$) will be a sphere of radius $r_0 = Nc\Delta t$. At instant $t_1 = \Delta t$, it is at $x = v_F \Delta t$ and it emits another front, which at t_N will have a radius $r_1 = 4c\Delta t$. This process continues in a similar fashion. At instant $t_N = N\Delta t$, F emits a front which, at this same instant, has zero radius.

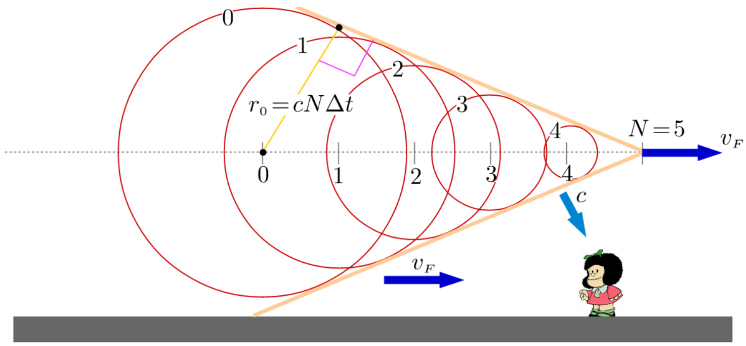


Fig. 8.17: Shock waves

As can be clearly seen in the figure, for any N, this succession of spherical waves gives rise to a conical envelope that concentrates all the wave fronts: This is the so-called **shock wave**, which travels behind the wave source itself. The overture half-angle of the cone θ fulfils $\sin\theta = cN\Delta t/(v_F N\Delta t)$, i.e.

$$\sin\theta = \frac{c}{v_F} \tag{8.60}$$

The shock wave travels at the velocity v_F of the source F while the wave travels at velocity c.

8.7 Diffraction

Consider a plane wave that strikes a wall that blocks its way except for one hole.If the hole is not very small, the wave passes through without altering its direction and, very roughly, it remains plane. If the hole is very small, the medium in the hole acts as a focus for spherical waves that propagate to the other side of the wall.

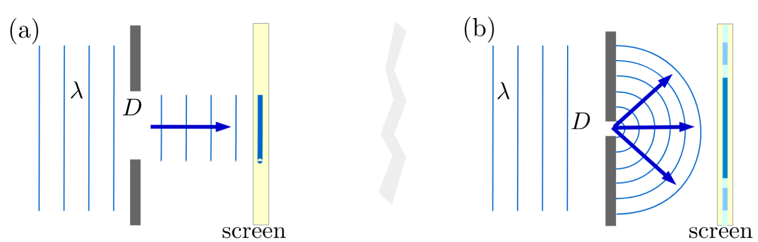


Fig. 8.18: Model for understanding diffraction

Figure 8.18 schematically illustrates these behaviours, which are the extremes of the set of phenomena known by the name **diffraction**. These occur when an object of dimensions comparable to the wavelength gets in the way of the wave.

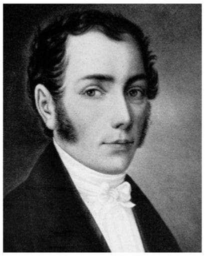


Fig. 8.19: Joseph von Fraunhofer (1787-1826) was a German optician

Fraunhofer diffraction

A very interesting diffraction situation can be observed when waves pass through a circular aperture of diameter D after originating at a distance much greater than

D (see Figure 8.20). This is called a **Fraunhofer diffraction**.

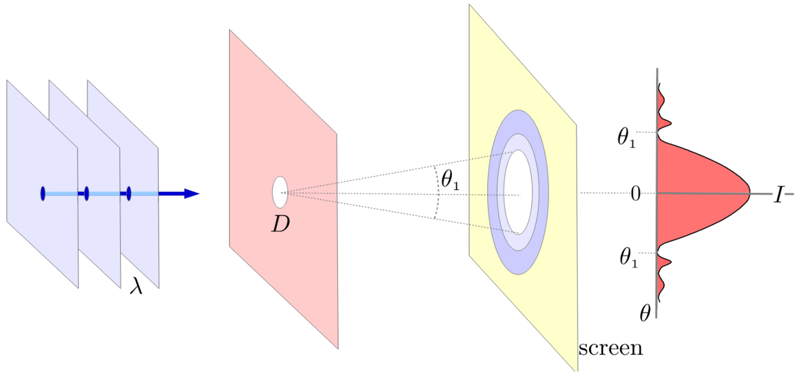


Fig. 8.20: Fraunhofer diffraction. The phenomenon is observed on a screen

A plane wave of wavelength λ is incident normal to the aperture. The intensity of the diffracted wave observed on the screen is a function of the angle θ and is distributed from a maximum value at $\theta = 0$ to a minimum for angle θ_1. This is the **Airy spot**. For $\theta > \theta_1$ values, the intensity goes through a series of weaker and weaker maxima and minima. In the Airy spot, determined by the θ_1 angle, 84% of the intensity diffracted by the aperture is concentrated. It can be shown and corroborated experimentally that the angle θ_1 is given by

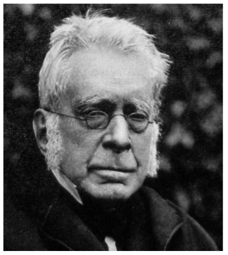
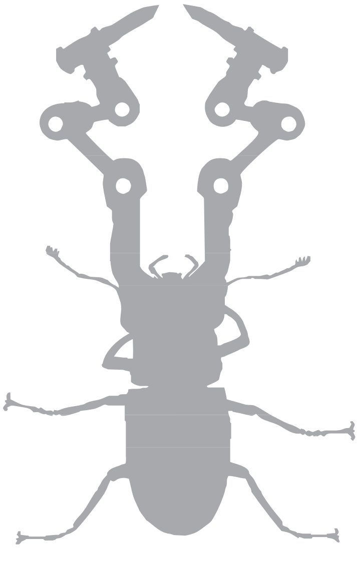


Fig. 8.21: Sir George Biddell Airy (1801-1892) was an English astronomer and mathematician

$$\sin \theta_1 = 1.22 \frac{\lambda}{D} \tag{8.61}$$



→ 9

9 Lagrange equations

Introduction

The only justification for including this chapter is to provide, for whoever wants it, a taste of analytical mechanics and its relationship with Newtonian mechanics through the general equation of dynamics, which was explained in Chapter 3. The Lagrangian equations of motion are derived from general equation of dynamics for systems with time-dependent geometric constraints. They are an extension to L degrees of freedom of Chapter 3's equations for conservative systems, deduced from their mechanical energy function. We will see how we can define the Lagrange function (also known as the Lagrangian) and how from it we find the L second order equations of motion that allow us to find the motion of the system. We do not go any further because our only objective is to provide a bridge between Newtonian mechanics and analytics in general, which many texts already cover in great detail.

9.1 Lagrange equations of the second kind

We will begin with the general equation of dynamics, explained in Section 3.9

$$\sum_{i=1}^{N}\left(\vec{F}_i - m_i\vec{a}_i\right)\cdot\delta\vec{r}_i = 0 \tag{9.1}$$

Fig. 9.1: Joseph Louis Lagrange (1736-1813) was an Italian mathematician, physicist and astronomer who later lived in Prussia and France

We will deal at most with geometric constraints that may be time-dependent. Let us assume that our constraints are expressed in parametric form, with the parameters $\{q_1, q_2 ... q_L\}$:

$$\vec{r}_i = \vec{r}_i(q_1, q_2 ... q_L, t) \tag{9.2}$$

where L is the system's number of degrees of freedom. We have

$$\delta\vec{r}_i = \sum_{a=1}^{L}\frac{\partial\vec{r}_i}{\partial q_a}dq_a \tag{9.3}$$

The term in (9.1) that contains the forces $\vec{F}_i$ can be written as

$$\sum_{i=1}^{N} \vec{F}_i \cdot \delta \vec{r}_i = \sum_{a=1}^{L} Q_a dq_a$$

where

$$Q_a = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_a} \tag{9.4}$$

are called **generalized forces**, which can always be expressed as a function of the parameters q_a, that is, the **generalized coordinates**. Note that the units of Q_a are not necessarily units of force (i.e., newtons); nor will all Q_a have the same units. The units of Q_a will depend on the units of q_a, and $Q_a\, dq_a$ will always have units of energy (i.e., joules).

The term for the acceleration in (9.1) can be written as

$$\begin{aligned} \sum_{i=1}^{N} m_i \vec{a}_i \cdot \delta \vec{r}_i &= \sum_{a=1}^{L} \sum_{i=1}^{N} m_i \frac{d\dot{\vec{r}}_i}{dt} \cdot \frac{\partial \vec{r}_i}{\partial q_a} dq_a \\ &= \sum_{a=1}^{L} \left(\frac{d}{dt} \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_a} - \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_a} \right) dq_a \end{aligned} \tag{9.5}$$

Taking into account that

$$\dot{\vec{r}}_i = \sum_{b=1}^{L} \frac{\partial \vec{r}_i}{\partial q_b} \dot{q}_b + \frac{\partial \vec{r}_i}{\partial t}$$

we have

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_a} = \frac{\partial \vec{r}_i}{\partial q_a} \tag{9.6}$$

and

$$\frac{\partial \dot{\vec{r}}_i}{\partial q_a} = \sum_{b=1}^{L} \frac{\partial^2 \vec{r}_i}{\partial q_b \partial q_a} \dot{q}_b + \frac{\partial^2 \vec{r}_i}{\partial q_a \partial t} = \frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_a} \tag{9.7}$$

With (9.6, 9.7), we can write (9.5) as

$$\sum_{a=1}^{L} \left(\frac{d}{dt} \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_a} - \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_a} \right) dq_a = \sum_{a=1}^{L} \left(\frac{d}{dt} \frac{\partial E_c}{\partial \dot{q}_a} - \frac{\partial E_c}{\partial q_a} \right) dq_a$$

so that the general equation of the dynamics (9.1) takes the form

$$\sum_{a=1}^{L} \left\{ Q_a - \left(\frac{d}{dt} \frac{\partial E_c}{\partial \dot{q}_a} - \frac{\partial E_c}{\partial q_a} \right) \right\} dq_a = 0$$

where E_c as the kinetic energy of the system, is a function of the generalized coordinates and time, i.e., $E_c(q_a, \dot{q}_b, t)$.

Finally, taking into account that dq_a are arbitrary displacements, we obtain the **Lagrangian equations of the second kind**

$$\frac{d}{dt}\frac{\partial E_c}{\partial \dot{q}_a} - \frac{\partial E_c}{\partial q_a} = Q_a \tag{9.8}$$

9.2 Lagrange equations

If the forces $\vec{F}_i$ are conservative, we have

$$\sum_{i=1}^{N} \vec{F}_i \cdot \delta\vec{r}_i = -dU = -\sum_{a=1}^{L} \frac{\partial U}{\partial q_a} dq_a$$

and therefore,

$$Q_a = -\frac{\partial U}{\partial q_a} \tag{9.9}$$

where U will be a function of q_a and perhaps of t, but not of $\dot{q}_a$. Without going through the forces $\vec{F}_i$, if the generalized forces Q_a satisfy (9.9) with $U(q_a, t)$, then the system has **potential forces** and U is its potential energy. If $U(q_a)$ does not depend on time, the generalised forces are conservative.

If we define the **Lagrange function** $\mathcal{L}$

$$\mathcal{L} = E_c - U \tag{9.10}$$

and we take into account that U does not depend on $\dot{q}_a$, we can write equations (9.8) for systems with potential forces, using the Lagrange function $\mathcal{L}$, in the form of **Lagrange equations**:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_a} - \frac{\partial \mathcal{L}}{\partial q_a} = 0 \tag{9.11}$$

If not all $\vec{F}_i$ are conservative, the Lagrangian equations take the form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_a} - \frac{\partial \mathcal{L}}{\partial q_a} = Q_a \tag{9.12}$$

where Q_a will be given according to (9.4) by taking into account only non-conservative forces or, in general, forces whose potential energy has not been included in the Lagrangian.

The Lagrangian equations have some important properties that will be discussed in the following.

Equivalent Lagrangians

If two Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ differ in a function of the form $\frac{d\Omega(q_a,t)}{dt}$, $\mathcal{L}' = \mathcal{L} + \frac{d\Omega}{dt}$, the associated Lagrangian equations are equal. The proof is immediate and is equivalent to proving the identity

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_a}\left(\frac{d\Omega}{dt}\right) - \frac{\partial}{\partial q_a}\left(\frac{d\Omega}{dt}\right) = 0$$

which can be done without using the Lagrange equations.

Energy conservation

If we define the function $\mathcal{H} = \sum_a \dot{q}_a \frac{\partial \mathcal{L}}{\partial \dot{q}_a} - \mathcal{L}$ and use the Lagrange equations, $\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{L}}{\partial t}$ is fulfilled. It is easy to see that, if the forces are conservative and the constraints do not depend on time, $\mathcal{L}$ will not depend on time; thus, the mechanical energy E will coincide with $\mathcal{H}$ and be conserved. The function $\mathcal{H}$ is the germ of a very important concept called the ***Hamilton function or Hamiltonian***.

Conservation and symmetry: Noether's theorem

Fig. 9.2: Emmy Noether (1882-1935) was a German mathematician

Given a continuous infinitesimal transformation of coordinates $q_a \to q_a + \delta q_a$, the Lagrangian becomes $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L}$

$$\delta\mathcal{L} = \sum_a \left(\frac{\partial \mathcal{L}}{\partial q_a}\delta q_a + \frac{\partial \mathcal{L}}{\partial \dot{q}_a}\delta \dot{q}_a\right)$$

If we *use the Lagrange equations*, we can write

$$\delta\mathcal{L} = \frac{d}{dt}\sum_a \frac{\partial \mathcal{L}}{\partial \dot{q}_a}\delta q_a$$

➔ An infinitesimal transformation $q_a \to q_a + \delta q_a$ is a **symmetry transformation** if it leaves the form of the equations of motion invariant by transforming the Lagrangian into an equivalent one, $\delta\mathcal{L} = \frac{d\delta\Omega}{dt}$, *without using the Lagrange equations*. We can, therefore, state a theorem of great importance in modern physics:

➔ ***Noether's Theorem***. *If the Lagrangian $\mathcal{L}$ has the symmetry $q_a \to q_a + \delta q_a$ some quantities are conserved (they remain constant when the system moves according to the equations of motion)*

$$\sum_a \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_a} + \frac{\partial \Omega}{\partial q_a}\right)\delta q_a = \text{constant} \tag{9.13}$$

The simplest case is when the Lagrangian does not depend on any of the coordinates q_a, for example q_1. We can say that the Lagrangian is symmetric with respect to transformations $q_1 \to q_1 + \delta q_1$ and we have a conserved quantity

$$p_1 = \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \text{constant} \tag{9.14}$$

Hamilton's principle

Assuming we already know the Lagrangian function $\mathcal{L}(q_a, \dot{q}_a, t)$, the Lagrangian equations (9.11) can be obtained from Hamilton's principle. To state this principle, one must first define the **action functional** S. If $q_{a1}(t_1)$ and $q_{a2}(t_2)$ are, respectively, the initial and final configurations of the system, the action functional is defined as a function of the trajectories $q_a(t)$ going from $q_{a1}(t_1)$ to $q_{a2}(t_2)$ in the form

$$S[q_a(t)] = \int_{t_1}^{t_2} \mathcal{L}(q_a(t), \dot{q}_a(t), t)\, dt \tag{9.15}$$

Fig. 9.3: William Rowan Hamilton (1805-1865) was a British-Irish mathematician, physicist and astronomer

➔ ***Hamilton's principle****. Given a system with Lagrangian $\mathcal{L}(q_a(t), \dot{q}_a(t), t)$, the trajectory that the system will follow from $q_{a1}(t_1)$ to $q_{a2}(t_2)$ is an extremum of the functional $S[q_a(t)]$ with respect to the trajectories passing through these two configurations:*

$$\frac{\delta S[q_a(t)]}{\delta q_a(t)} = 0 \tag{9.16}$$

Hamilton's principle is also known as the **principle of least action** and is used in many situations in modern physics.

Let us now look at some simple, one-degree-of-freedom examples that can be treated with the methods in this chapter.

Problem 9.2.1. In the figure, we can see a pendulum formed by a mass m tied to the end of a rod that is rigid and of negligible mass. The hand takes the other end of the rod and moves it horizontally $x_0(t)$ with respect to its unforced position $x_0 = 0$. Find the Lagrangian and the equation of motion of the pendulum and provides details for small oscillations in the following cases:

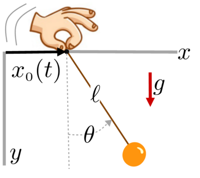


Figure for Problem 9.2.1

a) The pendulum oscillates without friction.

b) There is viscous friction, which we can express as a force $-b\vec{v}$, where $\vec{v}$ is the velocity of the mass m.

Solution

It is a constraint system. If the position of the mass m is $\vec{r} = (x, y)$, the constraint is $(x - x_0(t))^2 + y^2 = \ell^2$. If $x_0 = 0$, we have the usual pendulum.

The system has one degree of freedom, which we will represent with the angle θ. We have $\vec{r} = (x_0 + \ell \sin\theta, \ell\cos\theta)$ and $\vec{v} = \dot{\vec{r}} = (\dot{x}_0 + \ell\dot{\theta}\cos\theta, -\ell\dot{\theta}\sin\theta)$

The kinetic energy is $E_c = \frac{1}{2}m\left[(\dot{x}_0 + \ell\dot{\theta}\cos\theta)^2 + (-\ell\dot{\theta}\sin\theta)^2\right]$ and the potential energy is $U = -mg\ell\cos\theta$. Thereford, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta + m\dot{x}_0\dot{\theta}\ell\cos\theta \tag{1}$$

The equations of motion can be found from (1)

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} - \frac{\partial\mathcal{L}}{\partial\theta} = m\ell^2\ddot{\theta} + mg\ell\sin\theta + m\ddot{x}_0\ell\cos\theta = 0$$

If we consider small oscillations, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, we obtain

$$\ddot{\theta} + \frac{g}{\ell}\theta = -\frac{\ddot{x}_0}{\ell}$$

If the forced motion $x_0(t)$ is harmonic, the mass m performs forced harmonic oscillations, as we have seen in Section 6.4, but without friction.

To include friction, expressed as a force $-b\vec{v}$, (9.4) gives us

$$Q = -b\dot{\vec{r}}\cdot\frac{\partial\vec{r}}{\partial\theta} = -b\ell(\dot{x}_0\cos\theta + \ell\dot{\theta}) \tag{2}$$

We now obtain the Lagrange equations according to (9.12), that is,

$$m\ell^2\ddot{\theta} + mg\ell\sin\theta + m\ddot{x}_0\ell\cos\theta = -b\ell(\dot{x}_0\cos\theta + \ell\dot{\theta})$$

which, for small oscillations, is

$$\ddot{\theta} + \frac{b}{m}\dot{\theta} + \frac{g}{\ell}\theta = -\frac{\ddot{x}_0}{\ell} - \frac{b}{m}\frac{\dot{x}_0}{\ell} \tag{3}$$

As we can see, the friction affects the behaviour of the forced term. As described in Section 6.4, it will be a forced harmonic motion (FHM) only if the right-hand term of (3) has the form $B\sin(\Omega t + \theta_0)$, i.e., if the motion of x_0 fulfils

$$\ddot{x}_0 - \frac{b}{m}\dot{x}_0 = \ell B\sin(\Omega t + \theta_0)$$

Essentially, this means that x_0 has the form

$$x_0(t) = \frac{mB}{\Omega(\Omega^2m^2 + b^2)}\left(b\cos[\Omega t + \theta_0] - m\Omega\sin[\Omega t + \theta_0]\right) \qquad \blacksquare$$

Problem 9.2.2. In the figure, we can see a forced system formed by a mass m tied to the end of a spring of negligible mass. The hand takes the other end of the spring and moves it vertically so that it displaces by $x_0(t)$ the equilibrium position of the system $x_0 = 0$. Find the Lagrangian and the equation of motion for the following cases:

a) The system oscillates without friction,

b) There is viscous friction, which we can express as a force $-b\vec{v}$, where $\vec{v}$ is the velocity of the mass m.

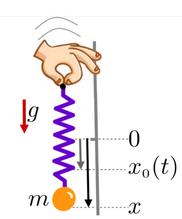


Figure for Problem 9.2.2

Solution

In this case, it is not a system with constraints. The net force on the mass m is $\vec{F} = -k(x - x_0(t))\hat{\imath}$. It is not a conservative force, since it depends on time, but it is a potential force: $U = \frac{1}{2}k(x - x_0(t))^2$. The kinetic energy is $E_c = \frac{1}{2}m\dot{x}^2$. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(x - x_0(t))^2 \tag{1}$$

The Lagrange equations are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = m\ddot{x} + kx - kx_0(t) = 0$$

The canonical equation is

$$\ddot{x} + \frac{k}{m}x = \frac{k}{m}x_0(t)$$

If we take into account the viscous friction, we have $Q = -b\dot{x}$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{\partial \mathcal{L}}{\partial x} = m\ddot{x} + kx - kx_0(t) = -b\dot{x}$$

The canonical equation is, in this case,

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{k}{m}x_0(t)$$ ■

→ 1P

1 Problems and questions

Problem 1.5.2. The following situation is to be studied: A solid ball moves down an inclined plane at an angle α to the horizontal. As far as we know, the ball falls with an acceleration of

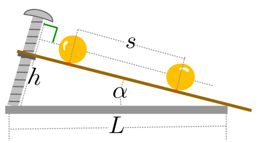


Figure for Problem 1.5.2

$$a = g\ \sin\alpha = g\frac{h}{L} \tag{1}$$

such that the motion is uniformly accelerated

$$s = \frac{1}{2}at^2 \tag{2}$$

However, the acceleration corresponding to the motion of the centre of the ball cannot in fact be that given by (1), since the gravitational potential energy is not only converted into translational kinetic energy, but also into rotational kinetic energy around its centre. This can lead to the acceleration of the sphere being reduced by a factor of $Q < 1$:

$$a = Q\ g\frac{h}{L} \tag{3}$$

It is this factor Q that we want to determine experimentally.

For a fixed and known path s, the ball's descent times are measured for various values of h. Using (2), the descent acceleration is determined in each case. Successive measurements are carried out by increasing h by a constant amount corresponding to one screw turn (thread pitch) d. If the initial height is h_0 and it is increased by n screw turns (with $n = 1, 2, 3, \ldots$), the height obtained for each value of n is

$$h_n = h_0 + nd \tag{4}$$

Data: Dimensions of the inclined plane $s = 35.3\,\text{cm}$; $L = 48.6\,\text{cm}$; screw thread pitch $d = 0.70\,\text{mm}$; acceleration of gravity $g = 9.81\,\text{m/s}^2$; the measurements can be seen in the table.

a) Find the acceleration a as a function of n , h_0 ,Q, d, L and g.

b) Represent the points (n, a) according to the measurements made and using (2).

n	0	1	2	3	4	5	6	7	8	9	10	11
t(s)	4.44	3.92	3.60	3.29	3.10	2.85	2.75	2.57	2.48	2.36	2.30	2.23

Table for Problem 1.5.2. For a fixed and known path s, the ball's descent times are measured for various values of n

c) Graphically find the line that best fits these points.

d) Find by **linear regression** the line that best fits these points.

e) Using the line of fit found in **d** and the expression found in **a**, deduce the values of the factor Q and h_0.

f) Find the error of the slope and the ordinate at the origin of the line of fit and then calculate the propagated error in the Q factor.

Question 1.7.1. A particle describes a trajectory characterised by a position vector $\vec{r} = \sin(t)\ \hat{\imath} + \cos(t)\hat{\jmath} + 2t\hat{k}$ (SI units). The radius of curvature of the trajectory is:

a) $5/2\,\text{m}$

b) $2/5\,\text{m}$

c) $5\,\text{m}$

d) $\sqrt{5}\,\text{m}$

e) $\frac{1}{\sqrt{5}}\,\text{m}$

Question 1.7.2. A particle describes a circular path of radius $1.5\,\text{m}$ and a tangential acceleration of $2t$ (SI units). At $t = 0$, the particle is at rest. The modulus of the acceleration of the particle at $t = 1.2\,\text{s}$ will be:

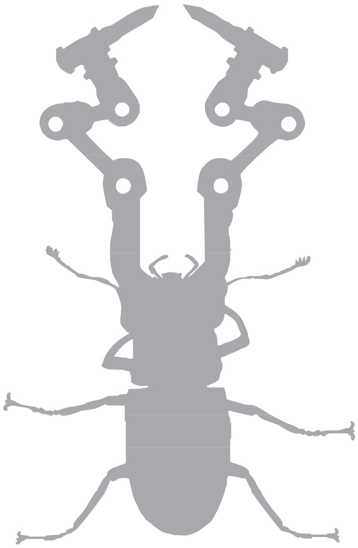
a) $1.38\,\text{m/s}^2$

b) $3.10\,\text{m/s}^2$

c) $1.44\,\text{m/s}^2$

d) $2.40\,\text{m/s}^2$

e) $2.77\,\text{m/s}^2$



→ 2P

2 Problems and questions

Question 2.1.1. On a $4\,\mathrm{kg}$ body moving in the direction of the x-axis, a force is applied in the same direction, as shown in the figure. At the initial instant, the body passes through $x = 0$ moving in the same direction as the force with a velocity of $3\,\mathrm{m\,s^{-1}}$. The velocity at position $x = 8\,\mathrm{m}$ will be:

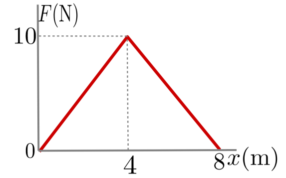


Figure for Question 2.1.1

a) the same as at $x = 0$

b) $\sqrt{29}\,\mathrm{m\,s^{-1}}$

c) $\sqrt{20}\,\mathrm{m\,s^{-1}}$

d) $7\,\mathrm{m\,s^{-1}}$

e) $\sqrt{11}\,\mathrm{m\,s^{-1}}$

Question 2.1.2. Newton's second law states that:

a) If we apply a variable force to a free body, it is directly proportional to the acceleration produced in the body.

b) Bodies with different masses are accelerated by the same force, with accelerations directly proportional to their masses.

c) Bodies with the same mass receive accelerations inversely proportional to the applied forces.

d) Force is neither created nor destroyed.

e) None of the above.

Question 2.1.3. The block in the figure is tied to a rope of $1.5\,\mathrm{m}$ and negligible mass at angle of 30° with the horizontal. The angular velocity is:

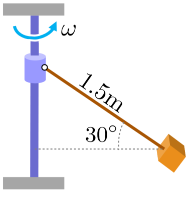


Figure for Question 2.1.3

a) $7.62\,\mathrm{rad/s}$

b) $4.62\,\mathrm{rad/s}$

c) $5.62\,\mathrm{rad/s}$

d) $6.62\,\mathrm{rad/s}$

e) $3.62\,\mathrm{rad/s}$

Problem 2.1.5. A $5\,\mathrm{kg}$ object is subjected to a force that varies with the position of the object $\vec{F} = \frac{2}{5}x\,\hat{\imath}$ (SI units). If it starts from $x_0 = 25\,\mathrm{m}$, what initial velocity v_0 will cause it to reach just $x = 0$?

Help. To integrate the equation of motion and find the velocity, multiply both members by the velocity and note that $v = \frac{dx}{dt}$

Solution: $v_0 = -7.07\,\mathrm{m/s}$

Problem 2.1.6. An object of $m = 4\,\mathrm{kg}$ is subjected to the action of two forces, $\vec{F}_1 = \hat{\imath} - 2\hat{\jmath}$ and $\vec{F}_2 = \hat{\imath} + \hat{\jmath}$ (SI units). Calculate the acceleration, velocity and position (vector and modulus) of the object at the instant $t = 3\,\mathrm{s}$ if at $t = 0$ it is at rest at the origin of coordinates.

Solution: $\vec{a} = \frac{1}{4}(2\hat{\imath} - \hat{\jmath})\ \mathrm{m/s^2}$; $a = \frac{\sqrt{5}}{4}\,\mathrm{m/s^2}$; $\vec{v}(3) = \frac{3}{4}(2\hat{\imath} - \hat{\jmath})\,\mathrm{m/s}$; $v(3) = \frac{3\sqrt{5}}{4}\mathrm{m/s}$; $\vec{r}(3) = \frac{9}{8}(2\hat{\imath} - \hat{\jmath})\ \mathrm{m}$; $r(3) = \frac{9\sqrt{5}}{8}\mathrm{m}$

Question 2.2.1. A tennis ball of mass $80\,\mathrm{g}$ normally hits a vertical wall. Both before and after the collision, the ball moves with a horizontal velocity of modulus $30\,\mathrm{m/s}$. The modulus of the impulse that the wall exerts on the ball is:

a) 0

b) $9.6\,\mathrm{N\,s}$

c) $2.4\,\mathrm{N\,s}$

d) $4.8\,\mathrm{N\,s}$

e) There are not enough data to calculate it

Question 2.2.2. A $60\,\mathrm{g}$ ball is dropped from a height of $2\,\mathrm{m}$. It bounces up to a height of $1.8\,\mathrm{m}$. Calculate how much the momentum changes during the impact with the ground.

a) $0.422\,\mathrm{kg\,m/s}$

b) $0.365\,\mathrm{kg\,m/s}$

c) $0.731\,\mathrm{kg\,m/s}$

d) $0.227\,\mathrm{kg\,m/s}$

e) $1.246\,\mathrm{kg\,m/s}$

Question 2.2.3. The position vector of a particle of mass $m = 2\,\mathrm{kg}$ is $\vec{r}(t) = (3t^4 + 1, 2t, 0)$ (SI units). It is true that:

a) The direction of the force is not constant.

b) The motion is rectilinear.

c) At $t = 0$, the angular momentum with respect to the origin is zero.

d) At $t = 0$, the momentum of the particle is $\vec{p} = (0\ ,\ 2\ ,\ 0)$.

e) The impulse applied by the force in the first second is $\vec{I} = (24\ ,\ 0\ ,\ 0)$.

Question 2.2.4. A $3\,\mathrm{kg}$ body moves rectilinearly according to $\vec{r}(t) = (1250 + 20t - 0.5t^2)\hat{\imath}$, for $t \geq 0$ (SI units). It is true that:

a) The initial velocity is $1250\ \hat{\imath}$.

b) The initial velocity is zero.

c) The initial momentum is $60\ \hat{\imath}$.

d) The force applied to the body is $3\ \hat{\imath}$.

e) The force applied to the body is $-1.5\ \hat{\imath}$.

Problem 2.2.3. A $2\,\mathrm{kg}$ particle is initially at the point $\vec{r}(0) = (0, 0)$, where it has a velocity of $\vec{v}(0) = (0, 2)\,\mathrm{m/s}$. If, from this point, a force $\vec{F}(t) = (8, -4)\,\mathrm{N}$ begins to act, calculate:

a) $\vec{r}(t)$, $\vec{v}(t)$ and $\vec{a}(t)$

b) Where will the particle be at $t = 5\,\mathrm{s}$?

c) The instant when the velocity along the y-axis will be zero.

d) Write the equation for the particle's trajectory.

Solution: a) $\vec{a}(t) = (4, -2)$; $\vec{v}(t) = (4t, 2 - 2t)$; $\vec{r}(t) = (2t^2, 2t - t^2)$; b) $\vec{r}(5) = (50, -15)$; c) 1 s ; d) $y = 2\left(\frac{x}{2}\right)^{1/2} - \frac{x}{2}$

Question 2.3.1. The trajectory of a particle of mass 2 kg is $\vec{r}(t) = -3t^2\,\hat{\imath} + (5t + 4)\,\hat{\jmath}$ (SI units). The angular momentum with respect to the origin of coordinates is (SI units):

a) $\vec{L}(t) = 6t(5t + 28)\,\hat{k}$

b) $\vec{L}(t) = 30t^2\,\hat{k}$

c) $\vec{L}(t) = (5t + 8)\,\hat{k}$

d) $\vec{L}(t) = (30t + 8)\,\hat{k}$

e) $\vec{L}(t) = 6t(5t + 8)\,\hat{k}$

Problem 2.3.3. A capsule of mass m makes a circular trajectory of angular velocity ω while tied to a rope of length ℓ, which has the other end fixed at height of $h = \ell/2$ to a dowel that can rotate without friction around the axis. If we increase h by $\Delta h = h/5$, what will the increase in $\Delta\omega$ be?

Solution: $\Delta\omega = \frac{11}{64}\omega$

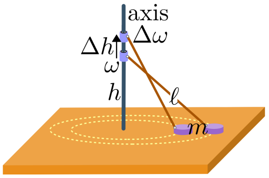


Figure for Problem 2.3.3

Problem 2.3.4. A small capsule of mass 250 g slides on a horizontal plate while it is tied to an inextensible thread that we grab up at the other end after it passes through a hole. The capsule describes a circular path of radius 500 mm. We pull the thread and consequently reduce the radius of the trajectory to 200 mm. If the initial velocity of the capsule is 5 m/s, determine its final velocity.

Solution: 12.5 m/s

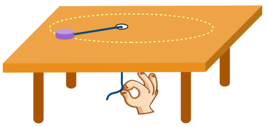


Figure for Problem 2.3.4

Problem 2.3.5. A point mass of 2 kg describes a trajectory of $x = t^3$; $y = t - 2t^2$; and $z = t^4/4$, where t is the time (SI units). Calculate at 2 s:

a) The velocity and acceleration vectors.

b) The momentum vector.

c) The angular momentum with respect to point $P = (7, -7.3)$.

d) The force acting on the point mass.

Solution: a) $\vec{v} = 12\,\hat{\imath} - 7\,\hat{\jmath} + 8\hat{k}$; $\vec{a} = 12\,\hat{\imath} - 4\,\hat{\jmath} + 12\hat{k}$; b) $\vec{p} = 24\,\hat{\imath} - 14\,\hat{\jmath} + 16\hat{k}$; c) $\vec{L}_{(P)} = 30\,\hat{\imath} + 8\,\hat{\jmath} - 38\,\hat{k}$; d) $\vec{F} = 24\,\hat{\imath} - 8\,\hat{\jmath} + 24\hat{k}$

Problem 2.3.6. A point mass attached to a rope of negligible mass and length ℓ oscillates in the vertical plane.

a) Find the equation of motion using the law of motion for angular momentum.

b) Specify the result for small oscillations.

Solution: a) $\ddot{\theta} = -\frac{g}{\ell}\sin\theta$; b) $\ddot{\theta} = -\frac{g}{\ell}\theta$

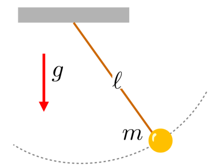


Figure for Problem 2.3.6

Question 2.4.1. A particle of 2 kg is moving in the direction of the x-axis and subject to a conservative force, the potential energy of which is $U = x^4 - 8x^2$ (SI units). It is true that:

a) The force is zero for $x = \pm 1$.

b) The acceleration at $x = 0$ is maximum.

c) The force acting on the particle has the expression $F = -x^4 + 8x^2$.

d) The force acting on the particle has the expression $F = -\frac{x^5}{5} + \frac{8}{3}x^3$.

e) The acceleration at $x = 1$ is $6\,\text{m/s}^2$.

Question 2.4.2. The figure shows a potential energy curve as a function of a variable x. Which of the following graphs representing the force as a function of this variable is compatible with the graph of $U(x)$?

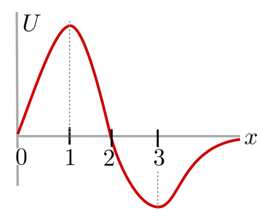
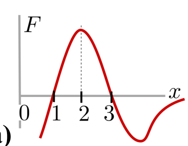
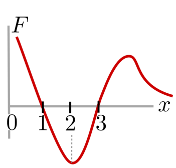


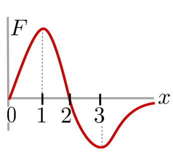
Figure for Question 2.4.2



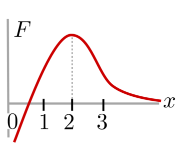
a)



b)



c)



d)

e) If the kinetic energy is not known, the force cannot be determined from $U(x)$.

Question 2.4.3. The force exerted on a 15 kg body moving in the direction of the x-axis varies with position, as shown in the figure. It is true that:

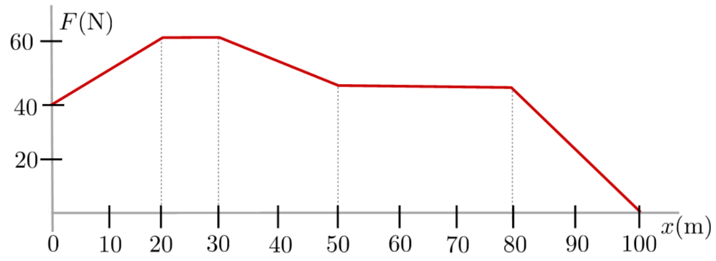


Figure for Question 2.4.3

a) When the body moves from the position $x = 0$ to $x = 50$ m, the potential energy decreases by 2600 J.

b) When the body passes from the position $x = 80$ m to $x = 100$ m, the kinetic energy decreases by 400 J.

c) In the section from $x = 50$ m to $x = 80$ m, the motion is uniform.

d) The maximum velocity reached by the body is in the section from $x = 20$ m to $x = 30$ m.

e) None of the above four answers are true.

Question 2.4.4. A body of 0.15 kg mass moves along the x-axis in a force field whose potential energy U is as shown in the figure. We leave the body at point C with a velocity of 4 m/s in the positive direction of the x-axis. It is true that:

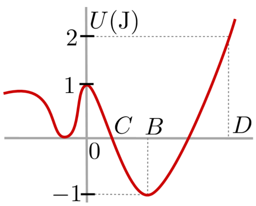


Figure for Question 2.4.4

a) The mechanical energy of the body is 0.3 J.

b) The velocity at point B is 3.4 m/s.

c) The velocity at $x = 0$ is zero.

d) At point B, the mechanical energy of the body is minimal.

e) The body cannot reach point D.

Question 2.4.5. A particle of $80\,\mathrm{g}$ moves along the x-axis under the action of a conservative force. Its potential energy is $U = x^4 - 2x^2$ (SI units). If the mechanical energy of the particle is $3\,\mathrm{J}$, its maximum velocity will be:

a) $2.4\,\mathrm{m/s}$

b) $10\,\mathrm{m/s}$

c) $8\,\mathrm{m/s}$

d) $6.79\,\mathrm{m/s}$

e) $3.8\,\mathrm{m/s}$

Question 2.4.6. A particle moves under the effect of a single conservative force. We can affirm that:

a) In any trajectory, the variation in potential energy between the initial and final positions of the trajectory is always zero.

b) If the particle describes a uniform circular motion, the variation in potential energy between two points of this trajectory is zero.

c) The increase in the mechanical energy of the particle can be positive or negative, depending on the sign of the work done by the force.

d) The potential energy of the particle increases if the work done by the force is positive.

e) None of the above four statements are correct.

Question 2.4.7. If we drop a stone from a castle tower and neglect the resistance of the air, it is true that:

a) The work of the gravitational force is zero.

b) The potential energy of the stone does not change.

c) The kinetic energy of the stone does not change.

d) The mechanical energy of the stone does not change.

e) The work of the gravitational force is negative.

Question 2.4.8. The potential energy of a particle of mass $5\,\text{kg}$ moving in the direction of the x-axis is shown in the figure. We can affirm that:

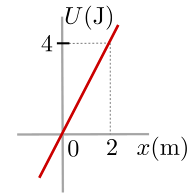


Figure for Question 2.4.8

a) The $x = 0$ position is of stable equilibrium.

b) The $x = 0$ position is of unstable equilibrium.

c) The acceleration at $x = 0$ is $-0.4\,\text{m/s}^2$.

d) The acceleration at $x = 2\,\text{m}$ is $-2\,\text{m/s}^2$.

e) The work done by the force when the particle passes from $x = 0$ to $x = 2\,\text{m}$ is $4\,\text{J}$.

Question 2.4.9. The force acting on a particle of mass $4\,\text{kg}$ is $\vec{F} = 6z\;\hat{k}$. If v is the modulus of the velocity, the mechanical energy of the particle is, except for a constant (all units in SI):

a) $E = 2v^2 - 3z$

b) $E = 4v^2 - 3z^2$

c) $E = 2v^2 - 3z^2$

d) $E = 2v^2 + 3z$

e) $E = 2v^2 - \frac{1}{3}z^3$

Problem 2.4.5. A particle of mass m moves in a force field of associated potential energy $U = 16 - (x^2 + y^2)$, in SI units.

a) Determine the equations of the equipotential surfaces in which $U = 0$, $U = 12$ and $U = 18$. Are they all possible? Draw the first one.

b) Find the locus of the points with maximum U. What is the maximum potential energy?

c) Find the analytical expression of the field of forces, the potential energy of which is the one given.

d) What force acts on the particle at the point $P_0 = (1, 1, 0)$? If it starts from P_0 at rest, what is the highest velocity it can reach at some later instant?

e) Calculate the work done by the force to take the particle from P_0 to $P = (22, 22, 0)$.

Solution: a) $x^2 + y^2 = 16$, $x^2 + y^2 = 4$, it does not exist; b) $x^2 + y^2 = 0$; $U_{max} = 16\,J$; c) $\vec{F} = 2x\hat{\imath} + 2y\hat{\jmath}$; d) $\vec{F} = 2\hat{\imath} + 2\hat{\jmath}$; $v = \infty$; e) $W = 966\,J$

Problem 2.4.6. A particle is subjected to a force $F = -kx + a/x^3$. Calculate the expression for the potential energy U.

Solution: $U = \frac{1}{2}kx^2 + \frac{a}{2x^2} + \text{ct}$

Problem 2.4.7. Given the force $\vec{F} = (3x - 4y)\hat{\imath} + (4x + 2y)\,\hat{\jmath}$, calculate:

a) The work done by $\vec{F}$ in carrying the particle from A to B; and the work done in one complete revolution along the ellipse in which a and b are, respectively, the major and minor semi-axes: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

b) Is the force $\vec{F}$ conservative? Why?

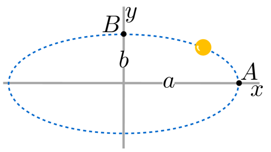


Figure for Problem 2.4.7

Note. The parametric form of an ellipse centred at the origin is

$$\vec{r}(\lambda) = (x, y) = (a\cos\lambda\ ,\ b\ \sin\lambda)$$

Solution: a) $\frac{2b^2 - 3a^2}{2} + 2\pi ab$, $8\pi ab$; b) no

Problem 2.4.8. The force acting on a particle is expressed as $\vec{F} = -ax^2\,\hat{\imath}$, where a is a constant. Calculate the potential-energy function taking $U = 0$ for $x = 0$ and plot the graph of U as a function of x.

Solution: $U = \frac{1}{3}ax^3$

Problem 2.4.9. A body of 2 kg mass moves along a horizontal plane xy where a force $\vec{F} = 2xy\,\hat{\imath} + x^3\hat{\jmath}$ acts.

a) Calculate the work of this force on the path from $A = (1, 1)$ to $B = (5, 9)$, following a straight line joining both points.

b) Calculate the work done between A and B when the path passes through $C = (5, 1)$ along the x and y axes while following, respectively, the paths AC and CB.

c) Can a potential energy be assigned to this force?

We substitute the previous force for another one $\vec{F} = -2x\,\hat{\imath}$. Now, we leave the body at point A with a velocity of 8 m/s and in the direction of the x-axis.

d) Determine the increase in potential energy at the AC displacement.

e) How far will the body travel while in motion?

f) At what velocity will it pass through the equilibrium position?

Note. Forces are expressed in N and distances in m.

Solution: a) $W = 453.3\,\text{J}$; b) $W = 1024\,\text{J}$; c) Non-conservative force, no potential energy can be defined; d) $\Delta U = 24\,\text{J}$; e) It reaches the point $(8.06, 1)\,\text{m}$; f) $v = 8.06\,\text{m/s}$.

Problem 2.4.10. A body of mass $2\,\text{kg}$ starts from rest and moves $10\,\text{m}$ without friction upwards in a plane inclined $30°$ from the horizontal. In addition to the weight, the three forces acting on the body are as shown in the figure. Calculate the total work done by the system of forces acting on the body and the power of these forces as a function of the distance x travelled.

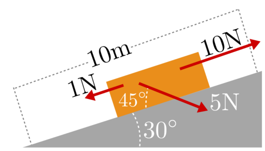


Figure for Problem 2.4.10

Solution: $W = 27.26\,\text{J}$; $P = 4.50\sqrt{x}$, in SI units

Problem 2.4.11. A $5\,\text{kg}$ object is subjected to a force that varies with the object's position, as shown in the figure. If it starts from rest at $x = 0$, what is its velocity at $x = 25\,\text{m}$ and at $x = 50\,\text{m}$?

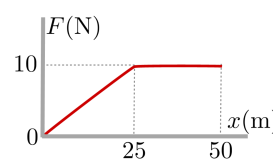


Figure for Problem 2.4.11

Solution: $v_{25} = 7.07\,\text{m}\,\text{s}^{-1}$; $v_{50} = 12.25\,\text{m}\,\text{s}^{-1}$

Question 2.5.1. We press a box of mass m against a vertical wall with a horizontal force of magnitude F. If the coefficient of friction with the wall is μ, the minimum value of F needed so that it does not slide down is:

a) mg

b) μmg

c) $\sqrt{1-\mu^2}mg$

d) $\frac{mg}{\mu}$

e) $mg\sqrt{\mu}$

Question 2.5.2. A particle of mass m moves along the x-axis and is attracted towards the origin of coordinates by a force $F = -k/x$, where k is a positive constant. If it starts from rest at point x_0, what is its velocity when it passes through point $x = x_0/2$?

a) $v = \sqrt{\frac{k}{m}}$

b) $v = \sqrt{\frac{2k}{m} \ln 2}$

c) $v = \sqrt{\frac{2k}{m}}$

d) $v = \frac{k}{m}\, e^{-\frac{kx_0}{m}}$

e) None of the above

Question 2.5.3. The figure represents a plane inclined at $30°$ with two identical springs of recovery constant $k = 20\,\mathrm{N/m}$, one at the top and the other at the bottom of the plane. A body with a weight of 2 N is released without initial velocity from the position shown in the figure, which is at the top where the spring is compressed for a distance x_1 (in metres). At the beginning of the movement, the body is 50 cm above the end of the lower spring. How much work (in joules) must be done by the frictional force between the body and the plane so that the maximum compression of the lower spring is also x_1?

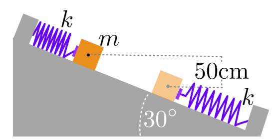


Figure for Question 2.5.3

a) $W_f = -(x_1 + 1)$

b) $W_f = -(x_1^2 + 1)$

c) $W_f = -\frac{(4x_1+2)}{20}$

d) $W_f = -20x_1^2$

e) We cannot know this because we do not know the natural length of the springs.

Question 2.5.4. A particle is subjected to a central force directed towards a point O. We can affirm:

a) The kinetic energy of the particle remains constant.

b) The trajectory of the particle is contained in a plane and its momentum remains constant.

c) The angular momentum of the particle with respect to point O is zero.

d) The angular impulse applied by the force with respect to point O is zero.

e) The trajectory of the particle will be circular.

Question 2.5.5. Which of the following forces are conservative:

A) The frictional force exerted on a sliding box.

B) The net force on a body from a spring, following Hooke's law.

C) The gravitational force

D) The force on a car from air resistance.

a) A, B and C

b) A and D

c) B and C

d) C and D

e) A, C and D

Problem 2.5.4. The body P in the associated figure has a mass of $m = 5\,\mathrm{kg}$ and rotates at an angular velocity of $20\,\mathrm{rev/min}$ about an axis EE' on a smooth conical (fixed) surface. The body is attached to the axis by a rope, of negligible mass and length $0.5\,\mathrm{m}$, parallel to the surface when P touches it. If $\alpha = 45°$, determine:

a) The reaction of the surface on the body.

b) The tension of the rope.

c) The angular velocity of the body at which it will move away from the surface.

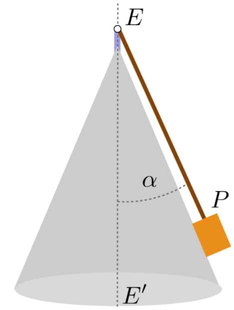


Figure for Problem 2.5.4

Solution: a) $29.6\,\mathrm{N}$; b) $40.13\,\mathrm{N}$; c) $5.26\,\mathrm{rad/s}$

Problem 2.5.5. A small body of mass $m = 5\,\mathrm{kg}$ falls from point A with no initial velocity down the track shown in the figure.

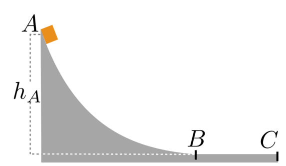


Figure for Problem 2.5.5

There is only friction on the BC section, the length of which is $4\,\mathrm{m}$, and the A point is situated at a height of $h_A = 2\,\mathrm{m}$. Knowing that the body stops at point C, calculate the coefficient of friction between the sliding body and the ground on section BC.

Solution: $\mu = 0.5$

Problem 2.5.6. We release a $2\,\mathrm{kg}$ block on a frictionless inclined plane $4\,\mathrm{m}$ from a spring of recovery constant $k = 100\,\mathrm{N/m}$. The plane is inclined at $30°$.

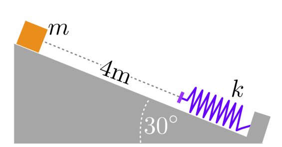


Figure for Problem 2.5.6

a) Calculate the maximum compression of the spring, assuming that it has no mass.

b) If there is friction and the coefficient of friction between the mass and the plane is 0.2, calculate the maximum compression.

c) In the last case, where there is friction, how far will the block travel along the plane after separating from the spring?

Solution: a) $0.989\,\mathrm{m}$; b) $0.783\,\mathrm{m}$; c) $1.54\,\mathrm{m}$

Problem 2.5.7. In order to deflect electrons in an oscilloscope, a constant electric field of $2000\,\mathrm{N/C}$ is used. Taking into account that, at the beginning, the electrons have a velocity of $10^6\,\mathrm{m/s}$, calculate the deflection when they have travelled $1\,\mathrm{cm}$ in the direction normal to the electric field.

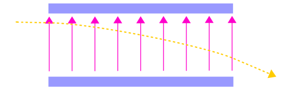


Figure for Problem 2.5.7

Solution: $1.76\,\mathrm{cm}$

Problem 2.5.8. In 2009, the Large Hadron Collider (LHC) was inaugurated in Geneva to study the properties of subatomic particles. Historically, this has been done in fog chambers where the particles leave trails of droplets due to condensation. Magnetic fields have been used to determine the charge of these particles and their mass. The force experienced by a charged particle passing through a magnetic field is $\vec{F} = q(\vec{v} \times \vec{B})$, where q is the charge of the particle, v its velocity and $\vec{B}$ is the magnetic field. Show that a charged particle in a magnetic field, uniform and normal to the plane of motion, describes a circular orbit of radius $R = \frac{mv}{qB}$, where m is the mass of the particle and v is the velocity.

Figure for Problem 2.5.8. Trajectories of subatomic particles within a magnetic field

Problem 2.5.9. From some clouds that are located about $2\,\mathrm{km}$ from the Earth's surface, calculate the speed at which raindrops would fall to the ground if there were no friction from the air. Repeat the calculation taking into account the air friction ($b = 10^{-4}\,\mathrm{kg/s}$) and that the mass of the droplets is $0.1\,\mathrm{g}$.

Solution: $197\,\mathrm{m/s}$; $9.8\,\mathrm{m/s}$.

Problem 2.5.10. A particle of mass $4\,\mathrm{kg}$ moves along an x-axis according to $x(t) = t + 2\,t^3$, (x in m and t in s). Calculate:

a) The kinetic energy as a function of time.

b) The acceleration of the particle and the force acting on it as a function of time.

c) The power supplied to the particle as a function of time.

d) The work done by the force from $t = 0$ to $t = 2\,\mathrm{s}$.

Solution: a) $(2 + 24t^2 + 72t^4)\,\mathrm{J}$; b) $12t\,\mathrm{m/s^2}$; $48t\,\mathrm{N}$; c) $(48t + 288t^3)\,\mathrm{W}$; d) $1248\,\mathrm{J}$

Problem 2.5.11. Two pads, A and B, of masses m_A and m_B, with pad-board friction coefficients μ_A and μ_B (dynamic = static friction), are placed on a horizontal board. We tilt the board:

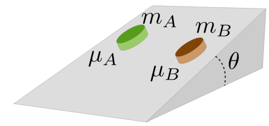


Figure for Problem 2.5.11

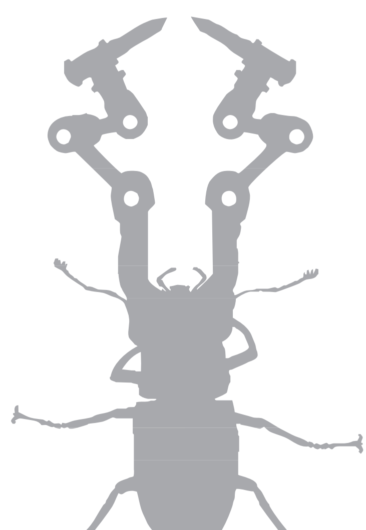
a) What is the necessary condition for A to initiate motion before B?

b) What is the necessary condition for the two bodies to slide together?

c) Given the above condition, calculate the value of θ that makes the motion of A and B uniform.

d) Calculate the acceleration of the motion when θ is greater than the value found in c).

Solution: a) $\mu_A < \mu_B$; b) $\mu_A = \mu_B$; c) $\tan\theta = \mu_A = \mu_B$; d) $a = g(\sin\theta - \mu\cos\theta)$



→ 3P

3 Problems and questions

Question 3.2.1. A homogeneous semicircular wire of mass m is clamped as shown in the figure by means of an O-joint. In equilibrium, θ is:

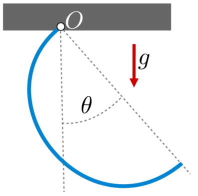


Figure for Question 3.2.1

a) $25.1°$

b) $23.0°$

c) $32.5°$

d) $39.5°$

e) It cannot be in equilibrium.

Question 3.2.2. A homogeneous wire is bent in the shape of a triangle, as shown in the figure. The coordinates of its CM are:

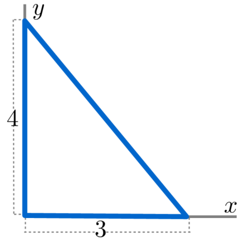


Figure for Question 3.2.2

a) $(1, 4/3)$

b) $(1, 3/2)$

c) $(3/2, 2)$

d) $(1, 2)$

e) None of the above are correct.

Question 3.2.3. The object in the figure consists of two wires of length L and linear density λ located in the xy plane; and a quarter of a circle of linear density 2λ located in the yz plane. The z-coordinate of the centre of mass is:

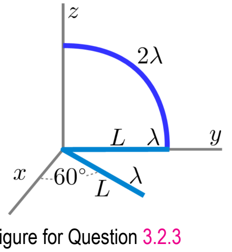


Figure for Question 3.2.3

a) $2L/(2+\pi)$

b) $2L/\pi$

c) $2L/(4+\pi)$

d) $L/2$

e) $L/4$

Question 3.2.4. What is the z-coordinate of the CM of the homogeneous quarter cone of height h in the figure?

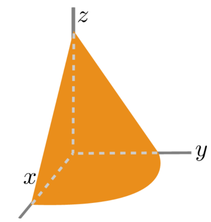


Figure for Question 3.2.4

a) $h/16$

b) $h/8$

c) $h/4$

d) $h/3$

e) $h/2$

Question 3.2.5. A homogeneous rectangular sheet with 4×8 cm sides has a circular hole of radius 1 cm. The centre C of the hole is at a distance b from the centre O of the rectangle and is located on the axis of symmetry, as can be seen in the figure. For the centre of mass of the sheet to be the point on the periphery of the hole, b must be:

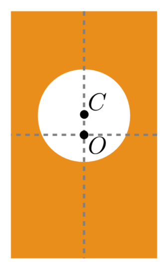


Figure for Question 3.2.5

a) 0.90 cm

b) 0.50 cm

c) 0.30 cm

d) 0.70 cm

e) 0.85 cm

Question 3.2.6. A 1500 kg car is moving westward at a speed of 20 m/s, and a 3000 kg truck is moving eastward at a speed of 16 m/s. Calculate the velocity of the centre of mass of the system:

a) 10 m/s westward

b) 4 m/s eastward

c) $2\,\text{m/s}$ eastward

d) $5\,\text{m/s}$ westward

e) $8\,\text{m/s}$ eastward

Question 3.2.7. The coordinates of the CM of the piece in the figure, which has the form of a circular sector of radius R, are:

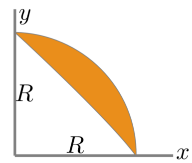


Figure for Question 3.2.7

a) $x = y = 0.50R$

b) $x = y = 0.33R$

c) $x = y = 0.58R$

d) $x = y = 0.62R$

e) $x = y = R$

Question 3.2.8. The piece in the figure has a uniform volume density. The upper part is a cylinder of radius r and height h, and the lower part is a hemisphere of radius $2r$. In order for the CM of the piece to be on the base of the cylinder, which of the following must be fulfilled?

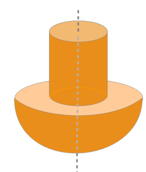


Figure for Question 3.2.8

a) $h = \sqrt{2}r$

b) $h = r/2$

c) $h = 2r$

d) $h = r$

e) $h = 2\sqrt{2}r$

Question 3.2.9. What are the coordinates of the centre of mass of the homogeneous sheet in the figure, which consists of a square and a circle?

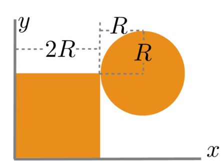


Figure for Question 3.2.9

a) $x_{CM} = R$; $y_{CM} = 1.44R$

b) $x_{CM} = R$; $y_{CM} = 2R$

c) $x_{CM} = 2.52R$; $y_{CM} = 1.76R$

d) $x_{CM} = 1.88R$; $y_{CM} = 2R$

e) $x_{CM} = 1.88R$; $y_{CM} = 1.44R$

Question 3.2.10. The distance from the centre of the base of the hat, made of a homogeneous sheet, to the centre of mass is:

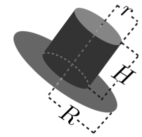


Figure for Question 3.2.10

a) $\frac{H^2 r}{R}$

b) $\frac{R^2 r + r^2 R}{H^2 + 2Rr}$

c) $\frac{H^2 r + r^2 H}{R^2 + 2Hr}$

d) $\frac{H^2 R + R^2 H}{r^2 + 2HR}$

e) $\frac{R^2 r}{H}$

Problem 3.2.3. Determine the CM of the homogeneous wire in the figure (SI units).

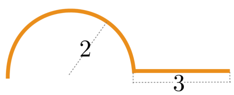


Figure for Problem 3.2.3

Solution. With the origin of (x, y) to the left of the beginning of the semicircle: $(3.13, 0.8)$

Problem 3.2.4. A homogeneous solid roller has the shape shown in the figure. The roller is built from a solid cylinder, from which we extract a hemispherical piece from one base and add it to the other base. Find its centre of mass if $a = 40\,\text{cm}$ and $d = 20\,\text{cm}$.

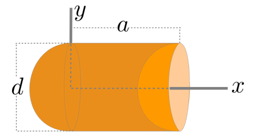


Figure for Problem 3.2.4

Solution: $x_{CM} = 40/3\,\text{cm}$

Problem 3.2.5. The figure shows a train consisting of three wagons, the masses of which are $m_A = 21 \times 10^3$ kg, $m_B = 18 \times 10^3$ kg and $m_C = 24 \times 10^3$ kg. The three wagons are connected by two chains of negligible mass.

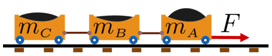


Figure for Problem 3.2.5

If a force of $F = 70\,\text{kN}$ is applied to the first wagon and parallel to the track and we assume a straight track with no friction, determine for the following conditions, first, the acceleration a with which the three wagons move and, second, the tensions in the two chains:

a) If the track is horizontal.

b) If the track is inclined upwards at an angle of $\theta = 15°$.

Solution:

a) $a = 1.11\,\text{m/s}^2$; $T_{AB} = 46.7\,\text{kN}$; $T_{BC} = 26.7\,\text{kN}$

b) $a' = -1.43\,\mathrm{m/s^2}$; $T'_{AB} = 46.7\,\mathrm{kN}$; $T'_{BC} = 26.7\,\mathrm{kN}$

Question 3.3.1. The necessary and sufficient condition for the momentum of a particle system to be conserved requires that:

a) The mechanical energy of the system is conserved.

b) The centre of mass of the system is at rest.

c) The external forces acting are conservative.

d) The resultant of the external forces is zero.

e) The internal forces of the system are conservative.

Problem 3.3.2. Imagine a 75 kg astronaut carrying a 5 kg tool (total = 80 kg) on a space walk outside the spacecraft, and the cable that connects him to the spacecraft breaks. Is there any way for him to get back? If he moves away from the spacecraft at a speed of $2\,\mathrm{m/s}$, would it be possible to return? Or would he stay in space forever and ever?

Solution. Launching the tool at a speed relative to the spacecraft $> 32\,\mathrm{m/s}$.

Problem 3.3.3. A 40 kg child stands at the end of an 80 kg, 2 m long platform. The child moves to the opposite end of the platform. Assume that there is no friction between the platform and the ground.

a) What is the displacement of the centre of mass of the system formed by the platform and the child?

b) How much does the child move relative to the ground? How much does the platform move relative to the ground?

c) If the child runs on the platform at a speed of $0.5\,\mathrm{m/s}$ (relative to the platform), how fast does the platform move?

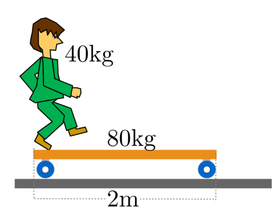


Figure for Problem 3.3.3

Solution: a) It does not move. b) The child moves $4/3$ m and the platform moves $2/3$ m in the opposite direction. c) $v = -1/6\,\mathrm{m/s}$.

Question 3.4.1. A system consists of two particles of the same mass m. At a given instant, one of them is at rest at point $(0, 0, 0)$ and the other, located at $(0, L, 0)$, is moving at a velocity of $\vec{v} = v\ \hat{\imath}$. The angular momentum of the system with respect to point $(0, L/2, 0)$ is:

a) $2mLv\,\hat{\imath}$

b) $-\frac{mLv}{2}\hat{k}$

c) $2mLv\,\hat{k}$

d) $\frac{mLv}{2}\hat{\imath}$

e) $mv\,\hat{k}$

Question 3.4.2. Two particles of masses $m_1 = 10\,\text{kg}$, $m_2 = 20\,\text{kg}$ move relative to an inertial reference frame according to $\vec{r}_1 = 2t^2\,\hat{\imath} + 3\hat{\jmath}$ and $\vec{r}_2 = 4t^2\,\hat{\imath} - 3\hat{\jmath}$. What is the angular momentum of the system relative to the origin?
Note: All units are expressed in SI

a) $\vec{L} = -600t\,\hat{k}$

b) $\vec{L} = -360t^2\,\hat{k}$

c) $\vec{L} = 12t\,\hat{k}$

d) $\vec{L} = 360t\,\hat{k}$

e) $\vec{L} = -12t\,\hat{k}$

Question 3.4.3. Two particles of masses $m_1 = 10\,\text{kg}$, $m_2 = 20\,\text{kg}$ move relative to an inertial reference frame according to $\vec{r}_1 = 2t^2\,\hat{\imath} + 3\hat{\jmath}$ and $\vec{r}_2 = 4t^2\,\hat{\imath} - 3\hat{\jmath}$. What is the value of the resulting moment of the forces relative to the origin?
Note: All units are expressed in SI

a) $\vec{M} = 12\,\hat{k}$

b) $\vec{M} = 600t\,\hat{k}$

c) $\vec{M} = 360\,\hat{k}$

d) $\vec{M} = -600\,\hat{k}$

e) $\vec{M} = -360t\,\hat{k}$

Problem 3.4.3. In figure skating, it is usual to increase angular speed by pulling in the arms. In a first approximation, we can assume that the arms have negligible masses due to the fact that they carry a weight on each hand and that the rest of the body is very close to the axis of rotation. Calculate the ratio between the final and initial angular velocity if the skater pulls in her hands from a distance of 1 m to 20 cm from her body.

Solution: $\omega_{\text{fi}}/\omega_{\text{ini}} = 25$

Problem 3.5.3. A child of mass $m = 35\,\text{kg}$ is at the end of a trolley of mass $M = 70\,\text{kg}$ which is initially at rest and can roll freely (see Figure). At a given instant, the child jumps with an exit velocity of $v_0 = 5\,\text{m/s}$ and at an angle of $\theta = 30^\circ$ to the ground, thus landing right at the other end of the trolley.

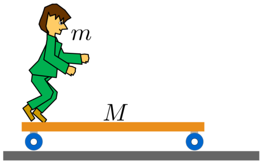


Figure for Problem 3.5.3

Neglecting any friction, determine:

a) The velocity at which the trolley moves backwards while the child is in the air during the jump.

b) The distance the child travels relative to the ground.

c) The distance travelled by the trolley and its length.

d) The speed of the trolley after the child has landed at the other end.

Solution: a) 2.17 m/s; b) 2.21 m; c) 1.11 m and 3.31 m; d) 0

Problem 3.5.4. A 50 kg block is in contact with a spring of recovery constant 300 N/m, of negligible mass and compressed to 0.2 m. The assembly is on a 75 kg wagon (the wheels are of negligible mass) and we release it. Determine the velocities of the block and the wagon from the instant when the block, sliding without friction on the wagon, loses contact with the spring.

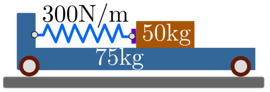


Figure for Problem 3.5.4

Solution

v: velocity of the block, positive to the right. V: velocity of the wagon, positive to the left.

Conservation of momentum: $0 = 50v - 75V$

Conservation of energy: $\frac{1}{2}300 \cdot 0.2^2 = \frac{1}{2}50v^2 + \frac{1}{2}75V^2$

Solving the system and taking the positive solution, which is in agreement with the fact that the spring is initially compressed:

$$v = 0.3795\,\text{m/s} \;\; ; \;\; V = 0.2540\,\text{m/s}$$ ■

Problem 3.6.2. A ball of mass 8 kg with a velocity of 5 m/s moves on a smooth horizontal floor and collides without friction with another ball of the same radius but a mass of 6 kg and at rest.

Determine the velocity and direction of the second ball if, as a consequence of the collision, the first ball is deflected 30° with respect to the original direction, assuming an elastic collision and that the floor and the balls are smooth.

Solution. Two possible solutions: $5.68\,\text{m/s}$ at -5.9° and $3.35\,\text{m/s}$ at -54.1°, the angles being with respect to the initial direction.

Problem 3.6.3. A ball of mass $m = 0.5\,\text{kg}$ is held by an inextensible rope of negligible mass and length $L = 1\,\text{m}$. We move the ball 90° from the equilibrium position and release it. On reaching the bottom of the trajectory, the ball collides inelastically with a block of mass $M = 3\,\text{kg}$, which is at rest on a surface. Knowing that the coefficient of restitution between the ball and the block is $e = 0.8$ and that the coefficient of friction between the block and the horizontal surface is $\mu = 0.2$, determine:

a) The velocity of the block and the ball after collision.

b) The work done by the frictional force until the block of mass M comes to rest.

c) The height reached by the ball after the collision.

Solution: a) m recoils at a velocity of $2.40\,\text{m/s}$ and M advances at a velocity of $1.14\,\text{m/s}$; b) -1.95J; c) $0.29\,\text{m}$

Problem 3.6.4. A $13\,\text{kg}$ block is at rest on a horizontal surface. A $400\,\text{g}$ mass of clay is thrown horizontally against it and sticks to it. The block and the clay slide $15\,\text{cm}$ along the surface. If the friction coefficient is 0.4, what is the initial velocity of the clay?

Solution: $36.4\,\text{m/s}$

Problem 3.6.5. An $11\,\text{kg}$ block explodes into three pieces of masses $m_1 = 4\,\text{kg}$, $m_2 = 6\,\text{kg}$ and m_3, with coplanar velocities $\vec{v}_1 = (v_{1x}, 3)$, $\vec{v}_2 = (4, v_{2y})$ and $\vec{v}_3 = (0, 2)$. Calculate v_{1x} and v_{2y}.

Solution: $v_{1x} = -6.00\,\text{m/s}$; $v_{2y} = -2.33\,\text{m/s}$

Problem 3.6.6. A $125\,\text{g}$ bird flying at $0.6\,\text{m/s}$ is about to catch a $5\,\text{g}$ bee flying perpendicularly at $15\,\text{m/s}$. What will be the speed of the bird after capturing the bee?

Solution: $0.816\,\text{m/s}$

Problem 3.6.7. We drop a $1\,\text{kg}$ sphere from a height of $1.8\,\text{m}$, which bounces on a $5\,\text{kg}$ plate, and reaches a height of $1.5\,\text{m}$. Determine:

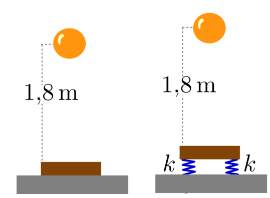


Figure for Problem 3.6.7

a) The coefficient of restitution in this case.

b) The maximum height the sphere would reach if the plate were supported by two springs of recovery constant $k = 2\,\text{kN/m}$ each and the impact were completely elastic.

c) Calculate the maximum deformation of the springs.

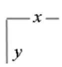
Solution: a) 0.913; b) $0.8\,\text{m}$; c) $0.07\,\text{m}$

Problem 3.6.8. A stone is dropped from the top of a $95\,\text{m}$ high tower and, one second later, a second identical stone is thrown up from the ground along the same vertical. Both collide head-on elastically at the midpoint of the tower. Determine:

a) The velocities of the stones immediately after the collision.

b) The height reached by the first stone after collision.

c) The height the second stone would have reached if there had been no collision.

Solution: 

a) $30.5\,\text{m/s}$ and $-12.32\,\text{m/s}$; b) $7.74\,\text{m}$; c) $55.225\,\text{m}$

Problem 3.6.9. A ball of mass $4\,\text{kg}$ is moving on a perfectly smooth horizontal floor at a velocity of $(20, 0)\,\text{m/s}$ relative to the reference system (x, y) fixed to the floor, and it collides with another ball of mass $6\,\text{kg}$ moving at a velocity of $(10, 10)\,\text{m/s}$.

a) If the two balls end up stuck together as a result of the collision, determine the (vector) velocity of the whole.

b) Calculate, in the case of a) above, the (translational) energy dissipated in the collision.

c) The two balls are smooth, have the same radius, and the collision is totally elastic. Find the two possible (vector) velocities of the second ball, if we know that the first ball leaves in the direction $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

Solution: a) $(14, 6)\,\text{m/s}$; b) $240\,\text{J}$; c) $(9.81, 2.19)\,\text{m/s}$ and $(19.39, 7.72)\,\text{m/s}$

Problem 3.7.2. The ***escape velocity*** associated with a planet or star is the minimum velocity for a body to escape the gravitational influence of that planet or star. As the object leaves the surface of the planet, reaches zero velocity as its distance approaches infinity. Show that this velocity is $v = \sqrt{2GM/R}$, where G is the gravitational constant, M is the mass of the planet and R is its radius.

A star with an escape velocity equal to the speed of light ($c = 3 \times 10^8\,\mathrm{m/s}$) is called a ***black hole***. Nothing can escape. Using Newtonian mechanics, calculate what radius the Sun ($M_{\text{Sun}} = 2 \times 10^30\,\mathrm{kg}$) would have to have to behave like a black hole.

Solution:$R_{\text{Sun}} = 2.93\,\mathrm{km}$

Problem 3.7.3. Show that the total energy of a body in circular orbit around a planet (the planet can be considered fixed) is half its potential energy.

Problem 3.7.4. Life on planet Earth as we know it will disappear in 10^9 years because the brightness of the Sun will grow until it eliminates any possibility of survival. To avoid this we have two possibilities (according to Korcyansky *et al.*): migrate to another planet or increase the radius of planet Earth's orbit from 1.5×10^{11} m to 2.2×10^{11} m (we approximate the Earth's orbit to a circle with the Sun fixed at its centre). The first possibility implies that the entire biomass of the planet (assumed to have mass $m_{\text{bio}} = 10^{20}$ kg) leaves the Earth's gravitational field, i.e., reaches escape velocity. From these data, which method is more energetically feasible (that is, it consumes the least energy): migrating all the biomass or changing the orbit of the planet?

Data: $M_{\text{Sun}} = 2 \times 10^{30}\,\mathrm{kg}$ and $R_{\text{Earth}} = 6 \times 10^6$ m.

> [Problem inspired by the article "Astronomical engineering: a strategy for modifying planetary orbits" published in the journal *Astrophysics and Space Science*, by D.G. Korcyansky *et al*. In turn, the author is inspired by calculations made by Carl Sagan in 1993 (Pollack & Sagan, 1993).]

Solution. Energy cost of changing the Earth's orbit: 8.4×10^{32} J. Energy cost of migrating away from the Earth: 6.57×10^{27} J

Problem 3.7.5. Gravitational acceleration on the Earth's surface can be calculated using data on the orbit of the Moon, just as Newton did. Calculate this value from the proposed data for the Moon: $T = 28$ days, $R_{\text{orbit}} = 3.8 \times 10^8$ m, $R_{\text{Earth}} = 6.37 \times 10^6$ m. Compare this result with the value of $g = 9.81\,\mathrm{m/s^2}$ known for the Earth's surface.

Solution: $g = 9.1\,\mathrm{m/s^2}$

Problem 3.7.6. Calculate the gravitational force of attraction that a newborn baby with a mass of 2 kg, feels to a midwife who has a mass of 70 kg and is standing at a distance $d = 0.5\,\text{m}$ from the baby. Calculate the gravitational attraction felt by the same child to the planet Mars, whose mass is $M_{\text{mars}} = 6.4185 \times 10^{23}$ kg at a distance of $d = 5.9 \times 10^7$ km from the baby. From these calculations, argue whether or not astrology is correct in asserting that the planets can define the character of people due to gravitational interaction.

Solution: $F_{\text{baby-midwife}} = 3.7 \times 10^{-8}\,\text{N}$; $F_{\text{baby-Mars}} = 7.92 \times 10^{-9}\,\text{N}$

Question 3.10.1. The particle in the figure, of mass 1 kg, is forced to pass through the wire without friction. At all times the resultant of the external forces is acting: $\vec{F} = -4y^3\,\hat{\jmath}$. If we release it without initial velocity at point A, what velocity will it have when passing through B? **Note**. All units are expressed in SI.

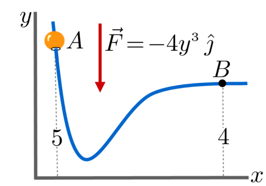


Figure for Question 3.10.1

a) $19.20\,\text{m/s}$

b) $27.17\,\text{m/s}$

c) $13.58\,\text{m/s}$

d) $11.04\,\text{m/s}$

e) $7.81\,\text{m/s}$

Problem 3.10.7. A bead of mass 3 kg falls with no initial velocity and slides without friction in a vertical plane along the guide in the figure. The spring to which it is attached has a recovery constant of $4\,\text{N/cm}$. The natural length of the spring is 60 cm. Find the velocity of the bead as it passes through B. Where does it stop?

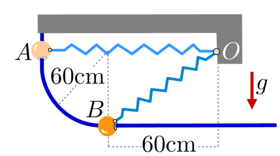


Figure for Problem 3.10.7

Solution: $v_B = 7.18\,\text{m/s}$. It stops at point C, $\overline{BC} = 1.72\,\text{m}$

Problem 3.10.8. A body of mass m, of small dimensions, is dropped from point A along the rail shown in the figure. If the body slides without friction and $h_A = 3R$, where R is the radius of the circumference, find:

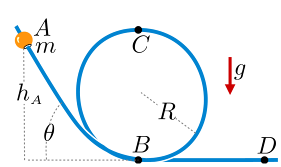


Figure for Problem 3.10.8

a) The force exerted by the body on the rail at the points B and C.

b) The value of the height of point A so that the force at point C is zero.

Solution: a) $F_B = m\frac{v_B^2}{R} + mg = 7mg$ and $F_C = m\frac{v_C^2}{R} - mg = mg$; b) $h'_A = \frac{5}{2}R$

Problem 3.10.9. A ball of mass 0.1 kg is strung on an ellipse-shaped guide with $a = \sqrt{2}$ and $b = \sqrt{3}$ being, respectively, the semi-axes of the ellipse: $\frac{x^2}{2} + \frac{y^2}{3} = 1$. In addition, there is a force field $\vec{F} = (3x - 4y)\hat{\imath} + (4x + 2y)\,\hat{\jmath}$. The ball leaves point A with a velocity $\vec{v}_A = 10\,\hat{\jmath}$ (see Figure). Calculate:

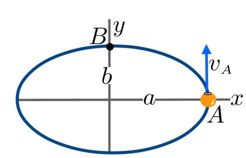


Figure for Problem 3.10.9

a) The velocity at which the ball passes through point B.

b) The velocity at which it passes through point A again.

Note 1: All quantities are given in SI. **Note 2:** The parametric equation of an ellipse is $\vec{r}(\lambda) = (a\cos\lambda, b\sin\lambda)$.

Solution: a) $v_B = 20.2\,\mathrm{m/s}$; b) $v'_A = 36.5\,\mathrm{m/s}$.

Problem 3.10.10. A reel of mass m is accelerated without friction by a force provided by a rope (of negligible mass) attached to a ball of mass m, which slides down a tube. A pulley (small and frictionless) and the tube are attached to a platform on which the reel moves. Together with the platform have a mass of $2m$ and slide along a horizontal guide. Find the velocity V of the platform as a function of the y-coordinate of the ball (see Figure) if, with everything stopped, we release the reel when the ball is at $y = 0$.

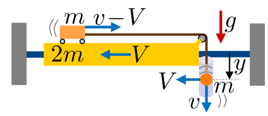


Figure for Problem 3.10.10

Note. The velocities in the figure refer to an observer at rest.

Solution: $V = \sqrt{\frac{1}{14}gy}$

Problem 3.10.11. A reel of mass m is accelerated without friction by a force provided by a spring (of negligible mass) attached to a platform of mass $2m$, which slides along a frictionless guide. If, when everything is stopped, we release the reel when the elongation of the spring is $x = L$, find the velocity v of the reel when the elongation of the spring is $x < L$.

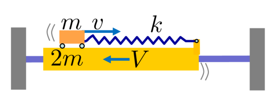


Figure for Problem 3.10.11

Note. The velocities in the figure refer to an observer at rest.

Solution: $v = \sqrt{\frac{2k}{3m}\left(L^2 - x^2\right)}$

Question 3.12.1. A homogeneous cylinder of radius R moves in a vertical plane along a horizontal surface by rolling and sliding. If, at a given instant, the velocity of the centre of mass is v to the right and that of the point of contact with the surface is $\frac{v}{2}$ also to the right, the modulus of the angular velocity of the cylinder at this instant is:

a) $\frac{2v}{3R}$

b) $\frac{3v}{4R}$

c) $\frac{v}{2R}$

d) $\frac{4v}{3R}$

e) $\frac{3v}{2R}$

Problem 3.13.3. Two particles of masses $m_1 = 3\,\text{kg}$ and $m_2 = 6\,\text{kg}$ are joined by a rigid bar of negligible mass. Initially, they are at rest and are subjected, respectively, to the action of the forces (expressed in N) $\vec{F}_2 = 3\hat{\jmath}$ and $\vec{F}_1 = 6\hat{\imath} - 6\hat{\jmath}$, as shown in the figure. Determine the CM and the momentum as a function of time.

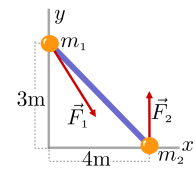


Figure for Problem 3.13.3

Solution: $\vec{R}_{CM} = \left(\frac{t^2}{3} + \frac{8}{3}\right)\hat{\imath} + \left(1 - \frac{t^2}{6}\right)\hat{\jmath}\ (\text{m})$ and $\vec{P}(t) = 6t\ \hat{\imath} - 3t\hat{\jmath}\ (\text{kg}\,\text{m}\,\text{s}^{-1})$

Problem 3.13.4. The square in the figure has the CM in the centre. If we apply **only** two forces of $2\,\text{N}$ as shown in the figure, will it rotate clockwise or counter-clockwise?

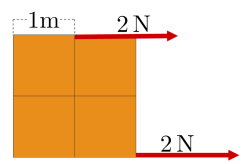


Figure for Problem 3.13.4

Solution: The square does not rotate.

Problem 3.13.5. A homogeneous bar of mass $1\,\text{kg}$ and $80\,\text{cm}$ long is placed on a vertical plane between two smooth parallel walls $40\,\text{cm}$ apart. The bar is attached to a spring of recovery constant constant $k = 50\,\text{N/m}$, as shown in the figure. Determine:

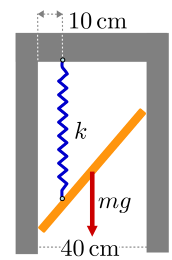


Figure for Problem 3.13.5

a)) The distance the bar has travelled.

b) The force that the spring exerts on the bar and the acceleration of the bar.

c) The forces exerted on the bar by the walls.

Solution: a) 39.2 cm; b) 19.62 N (upward); $9.81\,\text{m/s}^2$ (upward); c) equal and opposite direction, modulus 2.83 N

→ 4P

4 Problems and questions

Problem 4.2.1. The piece in the figure is made of two different wires. The horizontal one has a linear density which is double that of the semicircular part. What is the ratio between a and r if, when hanging the piece from point P, it remains in equilibrium, as shown in the figure?

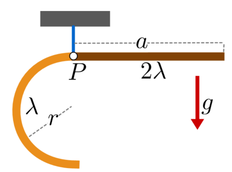


Figure for Problem 4.2.1

Solution: $a = \sqrt{2}\, r$

Question 4.3.1. The piston in the figure with surface area S is frictionless and in equilibrium under the action of the force exerted on it by the water. The gas inside the container has a pressure P. Knowing that the density of the water is ρ, and if P_a is the atmospheric pressure, the weight of the piston is:

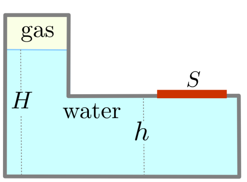


Figure for Question 4.3.1

a) $(H - h)\rho g S$

b) $(P - Pa)S + h\rho g S$

c) $(P - Pa)S + (H - h)\rho g S$

d) $H\rho g S$

e) $h\rho g S$

Question 4.3.2. The rigid body in the figure is made up of two homogeneous cubes, A and B, of density $900\,\mathrm{kg/m^3}$ and welded together so that their centres of mass are on the same vertical. The volumes of A and B are $1\,\mathrm{m^3}$ and $8\,\mathrm{m^3}$, respectively. The body is in equilibrium floating in water with cube B totally submerged, as can be seen in the figure. The centre of buoyancy in this equilibrium situation is at a distance from the free surface of the water, which is:

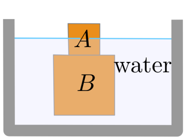


Figure for Question 4.3.2

a) $1.115\,\mathrm{m}$

b) 1.087 m

c) 1.057 m

d) 1.500 m

e) 1.322 m

Question 4.3.3. A cube of edge a and density $600\,\mathrm{kg/m^3}$ floats in a liquid with one third of the edge submerged. If we add a cylinder of the same density but with a volume of $2\,\mathrm{m^3}$ to the cube, we know that the cube will be completely submerged with its top base flush with the surface of the liquid. What is the length of the edge a?

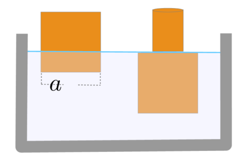


Figure for Question 4.3.3

a) 3 m

b) 1.5 m

c) 1 m

d) 0.5 m

e) 2.5 m

Question 4.3.4. In the vessel shown in the figure, the gauge pressure of gas 2 is $P_2 = 51300\,\mathrm{Pa}$. Knowing that the densities of the liquids are $\rho_1 = 3\,\mathrm{g/cm^3}$ and $\rho_2 = 1\,\mathrm{g/cm^3}$, the pressure of gas 1 will be ($g = 9.81\,\mathrm{m/s^2}$):

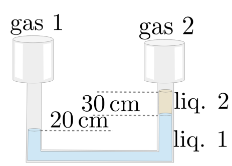


Figure for Question 4.3.4

a) 60129 Pa

b) 72240 Pa

c) −20340 Pa

d) 54328 Pa

e) 129354 Pa

Question 4.3.5. The 1.5 m wide body in the figure consists of two bodies of different densities, as shown in the figure. The centre of buoyancy is at a depth of:

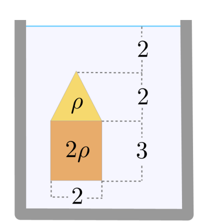


Figure for Question 4.3.5

a) 4.96 m

b) 6.05 m

c) $4.74\ \mathrm{m}$

d) $5.22\ \mathrm{m}$

e) $5.47\ \mathrm{m}$

Question 4.3.6. The spherical body in the figure is made up of two solid hemispheres, of which one density is twice than of the other. We tie it to a rope and immerse it halfway in a liquid, as shown in the figure. In this situation, the distance between the body's centre of buoyancy and centre of mass is:

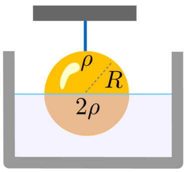


Figure for Question 4.3.6

a) $R/7$

b) $R/2$

c) $R/6$

d) $R/4$

e) $R/8$

Problem 4.3.4. We pour a 13 g drop of mercury into a beaker full of water 20 cm deep. Calculate how long it will take for the drop to reach the bottom of the beaker under the following conditions:

a) If we do not take into account the Archimedean buoyancy.

b) Taking into account the Archimedean buoyancy.

Data: $\rho_{\mathrm{Hg}} = 13 \times 10^3\ \mathrm{kg\,m^{-3}}$; $\rho_{\mathrm{water}} = 10^3\ \mathrm{kg\,m^{-3}}$

Solution: a) $a = g$, $t = 0.202\ \mathrm{s}$; b) $a = g(1 - \rho_{\mathrm{water}}/\rho_{\mathrm{Hg}})$; $t = 0.210\ \mathrm{s}$

Problem 4.3.5. The sewer network of two towns is linked by a series of underground conduits. One day the atmospheric pressure in town A is $P_A = 1.035 \times 10^5\ \mathrm{Pa}$ and that in town B is $P_B = 1.03 \times 10^5\ \mathrm{Pa}$. Recreate the diagram in the figure indicating in which well the height of the water will be higher and why. Calculate the difference in water height between the wells of town A and town B.

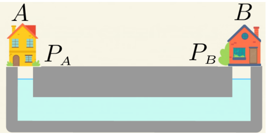


Figure for Problem 4.3.5

Solution: $0.051\ \mathrm{m}$

Problem 4.3.6. When a boat like the one in the figure is on the water, part of it is submerged. If we call the height of the water relative to the bottom of the boat h, will height h be greater if the boat is in fresh water or in salt water?

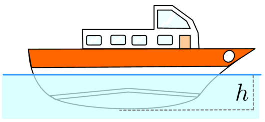


Figure for Problem 4.3.6

The boat in the figure has a total volume of 100 m³ and a mass of 2000 kg. Calculate what percentage of the volume of the boat is in the sea. If the same boat is in river water, calculate what percentage of the volume will be under water.

Data: $\rho_{\text{mar}} = 1.025 \times 10^3\,\text{kg}\,\text{m}^{-3}$, $\rho_{\text{riu}} = 10^3\,\text{kg}\,\text{m}^{-3}$

Solution. The percentage of boat submerged in sea water is 1.95% and in river water it is 2.00%

Problem 4.3.7. A container holds water and air, as shown in the figure. What is the gauge pressure, P_m, at points A, B, C and D?

Note: $P_m = P - P_{at}$.

Solution. $P_{mA} = 11760\,\text{Pa}$, $P_{mB} = P_{mC} = -2940\,\text{Pa}$, $P_{mD} = -17600\,\text{Pa}$

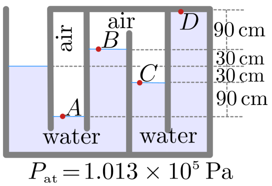


Figure for Problem 4.3.7

Question 4.5.1. The cylinder in the figure (of mass M) is kept in equilibrium by the force F of the rope (attached to the cylinder) and the friction between the cylinder and the wall (of coefficient μ). Indicate which of the following statements is correct.

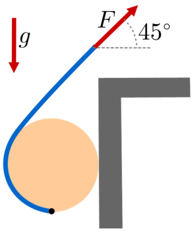


Figure for Question 4.5.1

a) In equilibrium it is verified that $F < Mg$.

b) In equilibrium it is verified that $F > Mg$.

c) There will only be equilibrium if $\mu > 1$.

d) If F is much greater than Mg, the frictional force and the weight of the cylinder have the same direction.

e) Equilibrium is impossible because there are three non-concurrent coplanar forces acting on the cylinder.

Question 4.5.2. The homogeneous triangular block in the figure of weight P has height h and base b. The coefficient of friction between the ground and the block is μ. Under conditions of sliding and imminent overturning, which of the answers is true?

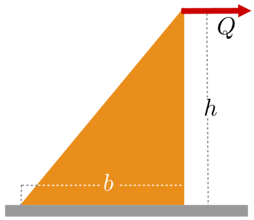


Figure for Question 4.5.2

a) $\mu = \frac{b}{3h}$

b) $\mu = \frac{2b}{3h}$

c) If $Q = 0$, the block can never be in equilibrium.

d) If $h > b$, the block will always rotate before sliding.

e) The above statements are wrong.

Question 4.5.3. We want to overturne a homogeneous block of weight W, without it sliding, by applying force F, as shown in the figure. The friction coefficient μ between the block and the horizontal floor must be:

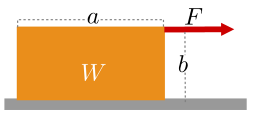


Figure for Question 4.5.3

a) $\mu < \frac{a}{2b}$

b) $\mu = \frac{a}{2b}$

c) $\mu > \frac{a}{2b}$

d) $\mu = \frac{F}{W}$

e) None of the above.

Question 4.5.4. Based on the figure, indicate which of these answers is correct:

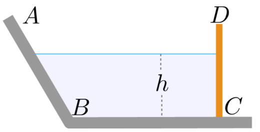


Figure for Question 4.5.4

a) The force of the water on the wall $\overline{AB}$ of the tank in the figure is horizontal and is applied at a height of $h/3$.

b) The buoyant force of the water on a body floating in equilibrium can have an arbitrary direction, depending on the shape of the body.

c) None of the other answers are correct.

d) If the barrier $\overline{CD}$ in the figure rests without friction on the bottom at C, it can be kept in equilibrium by applying an appropriate force F at a height of $h/2$.

e) The force of the water on the bottom $\overline{BC}$ of the tank has a horizontal component to the left.

Question 4.5.5. The wall of a reservoir has the trapezoidal cross-section shown in the figure. The height is h and the length in the transverse direction to the figure is ℓ. If g is the acceleration of gravity and ρ is the density of water, the modulus of the force exerted by the water on the wall is

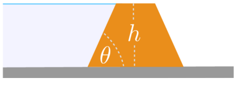


Figure for Question 4.5.5

a) $F = \frac{\rho g h^2 \ell}{2 \sin \theta}$

b) $F = \frac{\rho g h^2 \ell}{3}$

c) $F = \frac{\rho g h^2 \ell}{2}$

d) $F = \frac{\rho g h^2 \ell}{2 \cos \theta}$

e) It cannot be calculated from the data provided.

Question 4.5.6. A ball of mass $0.3\,\text{kg}$ is attached to point A on a wire with the shape of a semicircle of radius $50\,\text{cm}$ and centre C. The wire is homogeneous and has a mass of $0.2\,\text{kg}$. When the assembly is hung in the manner shown in the figure and once the equilibrium position is reached, the angle θ will be:

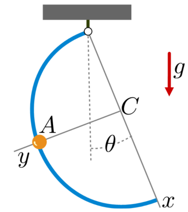


Figure for Question 4.5.6

a) $14.4°$

b) $17.6°$

c) $40.5°$

d) $32.5°$

e) $30.3°$

Question 4.5.7. A homogeneous rigid sphere of weight W and radius R is suspended from a wall by a wire of negligible mass and length $5/3R$, as shown in the figure. The junction A of the sphere with the wire is on the vertical passing through the centre of the sphere. There is no friction where the wall and the sphere are in contact at point B. If T is the tension of the wire and B is the reaction in the wall, we can affirm that:

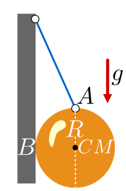


Figure for Question 4.5.7

a) $T = \frac{5}{3}W$

b) $B = \frac{4}{3}W$

c) $B = \frac{3}{4}W$

d) The wall reaction B is zero, taking into account that there is no friction.

e) The sphere cannot be in equilibrium in the position indicated in the statement.

Question 4.5.8. A thin homogeneous ring of radius R and weight P is placed on the inclined plane in the figure, supported by a cable $\overline{CB}$ running parallel to the inclined plane, with T being its tension. If μ is the coefficient of friction between the plane and the ring, and the ring is in a condition of imminent motion, indicate which of the following answers is true:

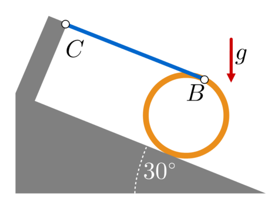


Figure for Question 4.5.8

a) $\mu = 0.29$

b) $\mu = 0.45$

c) $\mu = 0.84$

d) $\mu = \tan 60°$

e) $\mu = \tan 30°$

Question 4.5.9. The maximum tension supported by the cable in the figure is 600 N. The bar, joined at point A to the wall, has a weight of 800 N and a length of 8 m. To keep it in equilibrium in a horizontal position, the maximum distance from its CM to the end A must be:

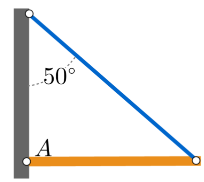


Figure for Question 4.5.9

a) 4 m

b) 3.51 m

c) 4.63 m

d) 3.25 m

e) 3.86 m

Question 4.5.10. Two horizontal forces of modulus $F = 160$ N are applied to a homogeneous cubic block of 3 m length and weight P. Their lines of action are parallel and displaced 0.3 m from the position of the centre of mass of the block, as shown in the figure. The coefficient of friction with the ground is 0.2. Indicate which of these statements is true:

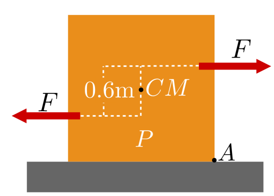


Figure for Question 4.5.10

a) In order for the block to be in equilibrium, the minimum value of P must be 192 N.

b) In order for the block to be in equilibrium, the minimum value of P must be 64 N.

c) In equilibrium, the normal component of the ground contact force passes through the centre of mass of the block.

d) If $P = 80$ N, the block is in equilibrium and the normal component of the ground contact force passes through point A.

e) If $P = 80$ N, the block is in equilibrium and the frictional force is 16 N.

Question 4.5.11. The system in the figure is in equilibrium. If the bar $\overline{AC}$ has negligible mass relative to 100 kg, the tension of the horizontal wire $\overline{BC}$ (in N) is:

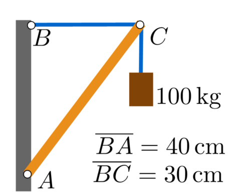


Figure for Question 4.5.11

a) 735.75

b) 981.00

c) 490.50

d) 1226.25

e) None of the above.

Question 4.5.12. If a beam is simply supported at ends A and B, supporting two vertical loads of $1000\,\mathrm{N}$, one applied at a midpoint and the other at support point A, it is true that:

a) The beam cannot be in equilibrium because the net moment with respect to B is not zero.

b) The beam is not in equilibrium because the net moment with respect to A is not zero.

c) The reaction at A is three times that at B.

d) The reaction at A is twice that at B.

e) The reaction at B is $1000\,\mathrm{N}$.

Question 4.5.13. A frictionless hinged gate at point A, width a, separates two liquids of densities ρ and $\rho/2$. What must the ratio be between heights H_1 and H_2 of the liquids so that the gate does not move?

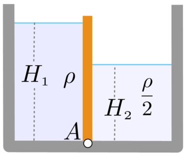


Figure for Question 4.5.13

a) $\frac{H_2}{H_1} = 2$

b) $\frac{H_2}{H_1} = \sqrt{2}$

c) $\frac{H_2}{H_1} = \sqrt[3]{2}$

d) $\frac{H_2}{H_1} = \frac{1}{2}$

e) $\frac{H_2}{H_1} = \frac{1}{\sqrt[3]{2}}$

Question 4.5.14. A square hinged frictionless gate of mass m is held in equilibrium at an inclined angle θ by a fluid of density ρ. In this situation, $\tan\theta$ is:

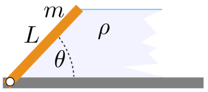


Figure for Question 4.5.14

a) $\frac{3m}{2\rho L^3}$

b) $\frac{3m}{\rho L^3}$

c) $\frac{m}{\rho L^3}$

d) $\frac{2m}{3\rho L^3}$

e) $\frac{m}{3\rho L^3}$

Question 4.5.15. The rope in the figure holds a gate of negligible weight, width L, and hinged without friction at the lower edge. What is the minimum tension of rope required to ensure that the gate can withstand the liquid of density ρ in the tank that is open at the top, as shown in the figure?

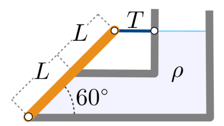


Figure for Question 4.5.15

a) $T = \frac{3}{4}\rho g L^3$

b) $T = \rho g L^3$

c) $T = \frac{1}{2}\rho g L^3$

d) $T = \frac{\sqrt{3}}{2}\rho g L^3$

e) $T = \frac{1}{3}\rho g L^3$

Question 4.5.16. The homogeneous ladder in the figure is supported by point A on the smooth wall and by point B on the rough floor. It is known to slide when $\theta \leq 30°$. The coefficient of friction with the ground is:

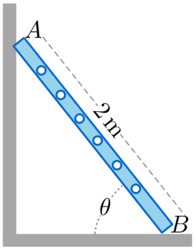


Figure for Question 4.5.16

a) $\mu = 0.766$

b) $\mu = 0.866$

c) $\mu = 0.666$

d) $\mu = 0.566$

e) $\mu = 0.966$

Question 4.5.17. Given the system in the figure, determine the coefficient of friction μ that causes body 2 to be in a condition of imminent downward motion if $m_1 = m$, $m_2 = 2m$ and $\theta = \frac{\pi}{3}$. The rope has negligible mass and the pulley shaft is frictionless.

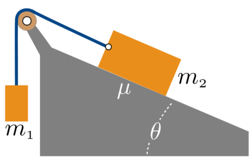


Figure for Question 4.5.17

a) 1.23

b) 1.46

c) 0.32

d) 2.46

e) 0.73

Problem 4.5.3. The homogeneous block in the position indicated in the figure has mass m and dimensions $a \times b \times b$. It is in equilibrium due to being submerged in a liquid of density ρ while tied to a rope at the centre of the lower edge. Find the mass m of the block.

Data: $a = 4\,\text{m}$, $b = 3\,\text{m}$ and $\rho = 10^3\,\text{kg/m}^3$.

Solution: $m = 12000\,\text{kg}$

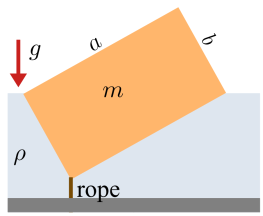


Figure for Problem 4.5.3

Problem 4.5.4. As shown in the figure, a uniformly weighted bar of length $2a$ and weight W rests on an empty, smooth hemisphere of radius r. Find the angle ϕ of equilibrium and the reactions at A and C.

Solution: $\cos\phi = \frac{a+\sqrt{a^2+32r^2}}{8r}$; $C = \frac{Wa}{2r}$, $A = W\ \tan\phi$

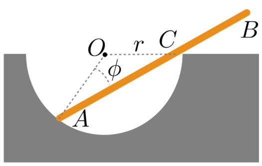


Figure for Problem 4.5.4

Problem 4.5.5. A homogeneous bar of 5 m and weight P is placed between two vertical walls 4 m apart. Wall A is smooth and the static friction coefficient between the bar and wall B is 0.9. An object of weight $3P$ can be fixed at different points of the bar. At what distance x from wall B can the object be fixed without losing its equilibrium?

Solution: $x > 3.78\,\text{m}$

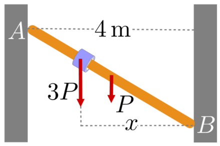


Figure for Problem 4.5.5

Problem 4.5.6. The end of a water channel is formed by a gate of negligible weight $\widehat{ABC}$, as shown in the figure. It is 1.2 m wide and hinged without friction at B. Calculate the ratio h/b for which the reaction at A is zero.

Solution: $\frac{h}{b} = \sqrt{3}$

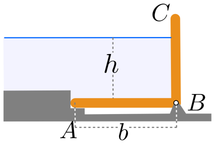


Figure for Problem 4.5.6

Problem 4.5.7. A uniform block A with a mass of 1 kg is placed on a rough horizontal plane of coefficient $\mu = 0.2$. This block is tied to a horizontal rope at height h. The rope passes through two massless and frictionless pulleys, C and D.

a) If $h = 0.3\,\text{m}$, determine the minimum value of the mass of body B that causes body A to lose equilibrium.

b) Do the same as in **a**, but if $h = 4\,\text{m}$.

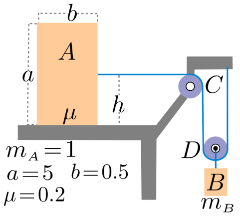


Figure for Problem 4.5.7. All units are expressed in SI

Solution: a) $m_B = 0.400\,\text{kg}$; b) $m_B = 0.125\,\text{kg}$

Problem 4.5.8. A 60 kg cabinet is mounted on wheels that can be locked to prevent them from rolling. The coefficient of friction is 0.3. If $h = 80\,\text{cm}$, determine the modulus of force P necessary to move the cabinet to the right under the following conditions:

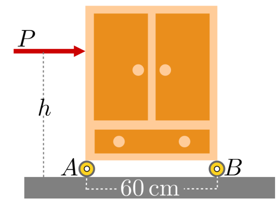


Figure for Problem 4.5.8

a) If all the wheels are locked.

b) If wheels B are locked and the wheels A can turn freely.

c) If wheels A are locked and wheels B can turn freely.

Solution: a) $P = 176.4\,\text{N}$; b) $P = 147.0\,\text{N}$; c) $P = 63.0\,\text{N}$

Problem 4.5.9. Determine the tension of a cable $\overline{AB}$ that supports a stick $\overline{BD}$ without slipping. The stick has a mass of 18 kg. Assume that there is no friction.

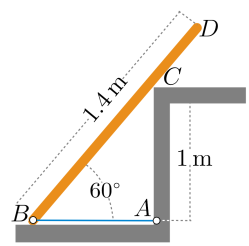


Figure for Problem 4.5.9

Solution: $T = 46.4\,\text{N}$

Problem 4.5.10. The rod in the figure is attached to the ground by the cable $\overline{AH}$. The mass of the rod is 1 kg and it is made up of two homogeneous sections of different densities ($\overline{BC}$ has twice the linear density of $\overline{AB}$). From end C hangs a 5 kg body. The system is balanced and the support at B is not smooth. Determine:

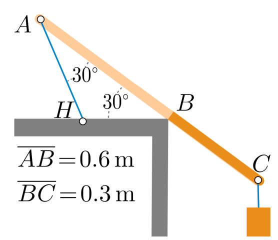


Figure for Problem 4.5.10

a) The tension of the cable.

b) The frictional force between the rod and the ground at B.

c) Would there be equilibrium if the coefficient of friction at B were 0.5?

Solution: a) $T = 40.3\,\text{N}$; b) $F = 64.3\,\text{N}$; c) No

Problem 4.5.11. A disc of radius R and mass M is on a horizontal surface, resting against a step of height $h = R/2$. We want the disc to go up the step by applying a force of modulus F to its axis (see Figure). Determine the minimum value of F for the disc to go up the step, as well as the modulus and direction of the force that the step exerts on the disc.

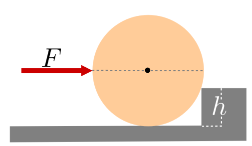


Figure for Problem 4.5.11

Solution: $F = Mg\sqrt{3}$; $N = 2Mg$; angle of N with the horizontal: 30°

Problem 4.5.12. A uniform rectangular plate $\overline{AB}$, of mass 1600 kg, separates two tanks containing water. The plate is hinged at the bottom, has a width of 3 m and is in equilibrium due to the action of the cable $\overline{BC}$. Determine the tension of this cable.

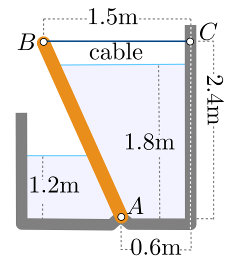


Figure for Problem 4.5.12

Solution: $T = 12512\,\text{N}$

Problem 4.5.13. A container is separated into two parts by a gate of height $\overline{AB} = 1.2\,\text{m}$, weight 5000 N, width $L = 3$ m and hinged without friction at point A. In the first part is a fluid of density $\rho_1 = 1.3$ g/cm^3 up to a height of $H_1 = 0.6$ m. In the second part is a fluid of density $\rho_2 = 1.8$ g/cm^3 up to a height of $H_2 = 1.5$ m. To keep the gate in equilibrium, a spring with a recovery constant $k = 3 \times 10^5$ N/m is used. Determine:

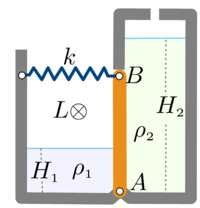


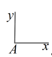
Figure for Problem 4.5.13

a) The force each fluid exerts on the gate.

b) Their respective points of application.

c) The force that the hinge exerts at point A of the gate.

d) The force exerted by the spring in order to maintain the gate's equilibrium, as well as the deformation it undergoes.

Solution: With the axes ,

a) $\vec{F}_1 = (6.887, 0) \times 10^3\text{N}$; $\vec{F}_2 = (-57.212, 0) \times 10^3\text{N}$

b) $\vec{r}_{F1} = (0, 0.20)$ m; $\vec{r}_{F2} = (0, 0.467)$ m

c) $\vec{A} = (29.201, 5) \times 10^3$ N; d) $\vec{F}_k = (21.117, 0) \times 10^3$ N ; $\Delta\ell = -0.070$ m

Problem 4.5.14. The tank in the figure is 3 m wide. With respect to C on the gate $\overline{CB}$, at what height h of the mercury will give rise to a moment in a clockwise direction due to the action of the two liquids of value 2×10^5 N m?

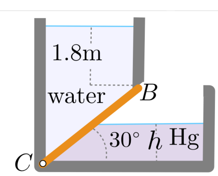


Figure for Problem 4.5.14

Data: Density of mercury: 13.6 g/cm^3. Distance $\overline{CB} = 2.5\,\text{m}$

Solution: $h = 0.23\,\text{m}$

Question 4.7.1. A particle of mass 20 kg receives the action of a conservative force of potential energy $U(x) = 5(x-2)^2 + 25$ (x in m and U in J). Which of the following statements is true?

a) The equilibrium position of the particle is $x = -2\,\text{m}$.

b) The potential energy associated with the equilibrium position is 25 J.

c) The equilibrium position of the particle is unstable.

d) If we leave the particle without velocity at the position $x = 0$, it will move in the negative direction of the x-axis.

e) The mechanical energy of the particle can be negative for certain values of x.

Question 4.7.2. Given a conservative system with one degree of freedom and of potential energy $U(x)$, which of the following statements is true?

a) To have stable equilibrium at $x = x_A$, it is necessary that at this position $U(x_A) = 0$.

b) The position $x = x_A$ cannot be of stable equilibrium if $\left.\frac{d^2U}{dx^2}\right|_{x=x_A} = 0$.

c) In order for the position $x = x_A$ to be of indifferent equilibrium, it is a necessary and sufficient condition that $\left.\frac{d^2U}{dx^2}\right|_{x=x_A} = 0$.

d) In a position of unstable equilibrium, the mechanical energy is maximum.

e) None of the above four statements are true.

Question 4.7.3. The frictionless hinged bar in the figure is homogeneous and weighs $P = 200$ N. The spring has recovery constant k and natural length $\ell_N = 0.5$ m. What must the value of k be so that $\theta = \frac{\pi}{4}$ is an equilibrium position?

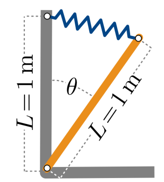


Figure for Question 4.7.3

a) $144.2\,\text{N/m}$

b) $288.4\,\text{N/m}$

c) $576.8\,\text{N/m}$

d) $96.3\,\text{N/m}$

e) $192.3\,\text{N/m}$

Problem 4.7.4. The figure shows the cross-section of a homogeneous ventilation door with a mass of 60 kg and hinged without friction at O. The door is controlled by a spring-loaded cable passing through the small frictionless pulley at A. The spring recovery constant is $160\,\text{N/m}$ and its deformation is zero when $\theta = 0$. Determine the equilibrium angle θ and analyse the equilibrium type.

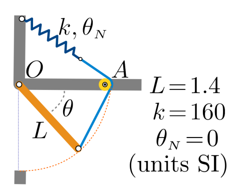


Figure for Problem 4.7.4

Solution: $\theta = 0.9202\,\text{rad} = 52.72°$; stable.

Problem 4.7.5. In the frictionless mechanism shown in the figure, A is a joint and C can slide along the wall. Find the equilibrium position y and the tension T of the spring $\overline{AC}$. The resting length of the spring is h, and k is its recovery constant.

Solution: $y = h + \frac{P}{2k}$; $T = \frac{P}{2}$

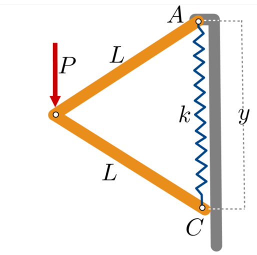


Figure for Problem 4.7.5

Problem 4.7.6. Given the potential $U(x) = x^4 - 4x^3 + 6x^2 - 4x + 10$ (SI units), is the point $x = 1$ stable, unstable or of indifferent equilibrium?

Solution: Stable.

Problem 4.7.7. Two bars $\overline{AB}$ and $\overline{BC}$ are each 0.6 m long and used to support a weight of 500 N. Knowing that there is equilibrium for $\theta = 30°$, determine the value of the recovery constant k for both springs. Assume that the weight of the bars and the friction are negligible. The springs are at their natural length when $\theta = 0$. Show that the equilibrium is stable. There is no friction between C and the ground. The upper end of the vertical spring can slide horizontally without friction; therefore, it always has the vertical position.

Solution: $k = 1029.6\,\mathrm{N\,m^{-1}}$

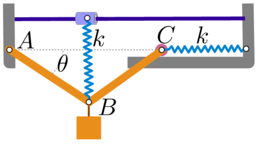


Figure for Problem 4.7.7

Problem 4.7.8. The figure represents a homogeneous window AB of weight 100 N, which can rotate, without friction, about A. It is held by a wire that is tied at B after passing through a frictionless pulley C, with the other end tied to a horizontal spring of recovery constant $k = 100\,\mathrm{N/m}$. The spring is at its natural length when $\theta = 30°$. If $\overline{AB} = \overline{AC} = 1.20$ m, find the angle θ of equilibrium of the system.

Solution: $\theta = 180°$ and $\theta = 52.7°$

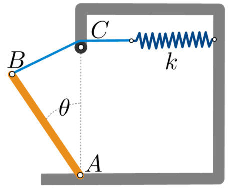


Figure for Problem 4.7.8

Problem 4.7.9. Four frictionless hinged rods of negligible mass form a rhombus $\widehat{ABCD}$, supported at A and subjected to the action of the given forces P and Q, as shown in the figure. Determine the equilibrium configuration of the system defined by the angle θ.

Solution: $P = Q \tan\theta$

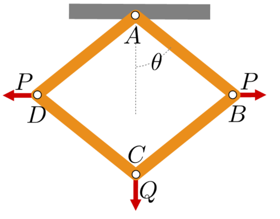


Figure for Problem 4.7.9

Problem 4.7.10. In the system shown in the figure, with no friction and the $\overline{AC}$ bar of negligible weight, find the angle θ of equilibrium.

Data: $a = 50$ cm, $k = 100\,\mathrm{N/m}$, natural length of the spring = 50 cm, and $P = 10$ N.

Solution: $\theta = 77.4°$ and $\theta = 180°$

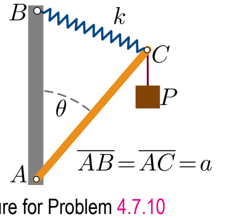


Figure for Problem 4.7.10

Problem 4.7.11. Determine the equilibrium position of the system in the figure. The bars have negligible weight, $\overline{AD}$ is the natural length of the spring and there is no friction.

Solution: $\sin\theta = 0$ and, if $\frac{P}{4ka} \leq 1$; $\cos\theta = \frac{P}{4ka}$

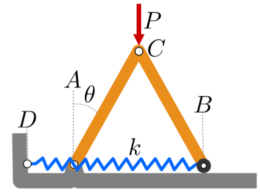


Figure for Problem 4.7.11

Problem 4.7.12. A vertical load W is applied to the mechanism in the figure. The spring recovery constant is k; its natural length corresponds to when $\overline{AB}$ and $\overline{BC}$ are horizontal. Assuming that the bars are weightless and there is no friction, find the expression relating θ, W, a and k, when the mechanism is in equilibrium.

Solution: $(1 - \cos\theta)\tan\theta = \frac{W}{4ka}$

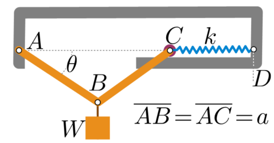


Figure for Problem 4.7.12

Problem 4.7.13. A small ball of mass m can move frictionlessly along the guide. In addition to being subjected to gravity, an inextensible rope of negligible mass passing through a small pulley O (free of friction) subjects it to the action of a spring of negligible mass and recovery constant k. The spring is relaxed when we unhook the end of the rope attached to the ball and we take it to point O.

a) Find the position (x, y) of equilibrium as a function of m, g, k and L.

b) Determine the type of equilibrium.

Solution: a) $x_{eq} = \frac{L}{2}\left(\frac{mg}{kL} + 1\right)$; b) stable.

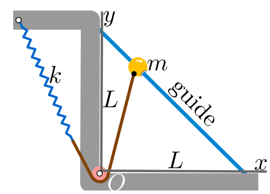


Figure for Problem 4.7.13

Problem 4.7.14. The uniform bar $\overline{AB}$ in the figure, with a weight of 150 N and a length of $L = 0.9$ m, can slide along a vertical guide without friction. The spring of recovery constant $k = 250$ N/m is attached to point B of the bar and has its natural length when $\theta = 0$. Neglecting the frictional forces with the ground, determine:

a) The equilibrium positions.

b) The type of equilibrium.

c) The reaction at A for the different equilibrium positions.

d) What would be the work done by the spring if the system were to move from position $\theta = 10°$ to $\theta = 30°$?

Solution: a) $\theta_1 = 0°$ and $\theta_2 = 48.2°$; b) θ_1 unstable and θ_2 stable; c) $A_1 = 150$ N; $A_2 = 75$ N; d) $W_{spring} = -1.79$ J

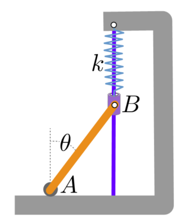


Figure for Problem 4.7.14

Problem 4.7.15. The system in the figure is in equilibrium. The bars have a negligible weight and are frictionless. Express the relationship between P and F as a function of the angle θ.

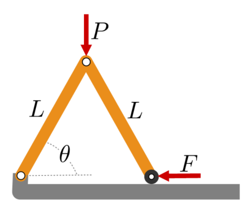


Figure for Problem 4.7.15

Solution: $\frac{P}{F} = 2\tan\theta$

Problem 4.7.16. The roly-poly toy in the figure has a symmetry of revolution and a hemispherical base, such it always returns to its initial vertical position after striking it. What condition must the assembly's centre of mass fulfil?

Figure for Problem 4.7.16

Solution: The CM must be below the base of the hemisphere.

Problem 4.7.17. The system shown in the figure consists of two springs of recovery constants $k = 100\,\mathrm{N/m}$ and a bar of mass 30 kg and length 2 m. The end A of the bar can move without friction in the horizontal direction x and the other end B can move on the vertical wall, which is also frictionless. Both springs have the same natural length corresponding to the vertical position of the bar. Calculate:

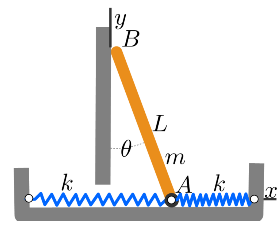


Figure for Problem 4.7.17

a) The values of x corresponding to the two equilibrium positions presented by the system.

b) The work of the spring on the right (W_{SR}) when the bar moves, in the negative direction of the x-axis, between these two positions.

c) The work of the gravitational force (W_G) in this displacement.

Solution: a) 0 and 1.8598 m; b) 172.92 J; c) -186.03 J

Problem 4.7.18. A small trolley of mass M (wheels of negligible mass) with an overload M_1 and M_2 is placed on an inclined plane and attached at one end to a spring (recovery constant k and natural length ℓ_n), while the other end is tied to a rope (always taut) passing through a pulley (of negligible mass and no friction), from which hangs a mass m.

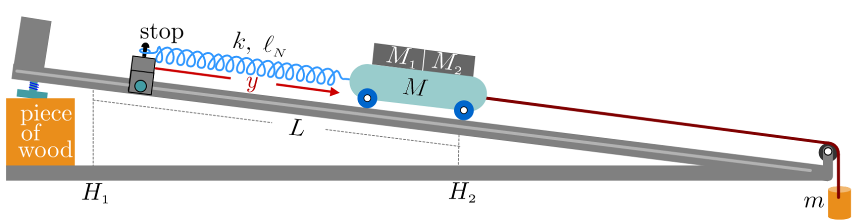


Figure for Problem 4.7.18

a) Write the expression for the mechanical energy of the system as a function of the y-coordinate (the energy may have an additive constant, independent of y).

b) Find the equilibrium position y_0.

c) Imposing conservation of energy, find the equation of motion.

d) Write the equation of motion as a function of the coordinate x, for which the equilibrium position is $x = 0$.

e) How does the trolley move if we release it from 5 cm downwards from the equilibrium position? What if we release it from 5 cm upwards?

f) What is the maximum distance (with respect to the equilibrium position) from which we can release the trolley without the rope becoming slack?

Data: $M = 0.487\,\text{kg}$;, $M_1 = 0.494\,\text{kg}$, $M_2 = 0.289\,\text{kg}$;, $L = 60\,c\text{m}$, $H_1 = 8.13$ cm, $H_2 = 1.22$ cm, $m = 105$ g, $k = 3.63\,\text{N/m}$ and $\ell_n = 9.5\,\text{cm}$

Solution:

a) $E = \frac{1}{2}(M+M_1+M_2+m)\dot{y}^2 - (M+M_1+M_2)gy\sin\alpha - mgy + \frac{1}{2}k(y-\ell_n)^2$
$\sin\alpha = 0.115$

b) $y_0 = \frac{((M+M_1+M_2)\ \sin\alpha + m)g + k\ell_n}{k}$.

c) $(M + M_1 + M_2 + m)\ \ddot{y} = ((M + M_1 + M_2) \quad \sin\alpha + m\ )g - k(y - \ell_n)$

d) $(M + M_1 + M_2 + m)\ \ddot{x} = -k\ x$

e) $x(t) = 0.05\cos\omega t$; $x(t) = -0.05\cos\omega t$; $\omega = \sqrt{\frac{k}{M+M_1+M_2+m}}$

f) distance $d = A < \frac{g}{\omega^2}$

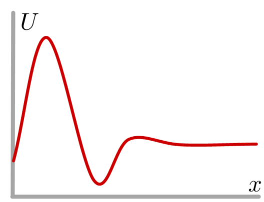


Figure for Problem 4.7.19

Problem 4.7.19. $U(x)$ is the potential energy function of a force as a function of the Cartesian x-coordinate.

a) Draw the attached graph by schematically indicating the modulus, direction of the forces at different points on the curve.

Using the graph, indicate:

b) The point where the force will be maximum, zero, attractive and repulsive.

c) The points of equilibrium and what these are points like.

If we assume that this curve represents the interaction potential between two atoms:

d) What energy will hold both atoms together?

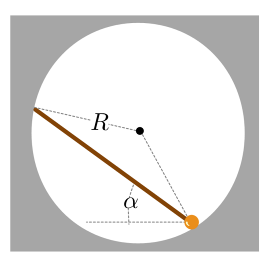


Figure for Problem 4.7.20

Problem 4.7.20. A homogeneous rod of mass m and length $R\sqrt{3}$ rests without friction inside a spherical cavity of radius R. A mass $m/2$ is fixed at one end of the rod. What is the angle α of equilibrium? What kind of equilibrium is it?

Solution: $\alpha = 30°$, stable

Problem 4.7.21. Determine the equilibrium positions of a homogeneous bar of length $L = 1\,\mathrm{m}$ and weight $10\,\mathrm{N}$, on which a spring of recovery constant $k = 10\,\mathrm{N/m}$ is acting. ℓ_N is the natural length of the spring, and the contacts with the wall and floor are smooth.

Solution: $x = 0$ and $x = 0.866\,\mathrm{m}$

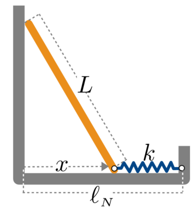
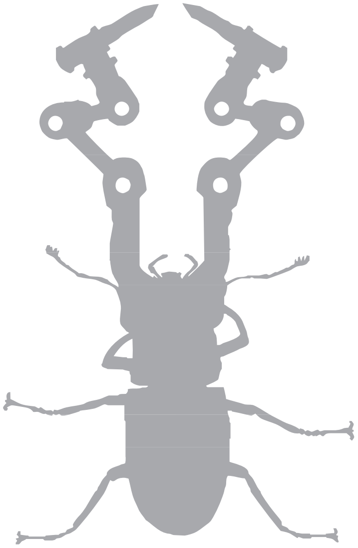


Figure for Problem 4.7.21



→ 5P

5 Problems and questions

Question 5.2.1. Which of the following statements is generally true about rigid body kinematics?

a) If a rigid body is in pure translational motion, no point on the body can describe a curvilinear motion.

b) Two points on the same body can have different angular velocities.

c) The kinematic condition of rigidity applies only to plane motion.

d) The relative velocity of two points on a moving body can never be zero.

e) All of the above statements are false.

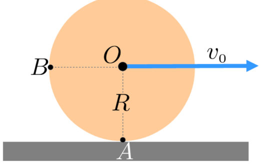


Figure for Question 5.2.2

Question 5.2.2. The figure shows a disc of radius R rolling on a flat surface without sliding. The velocity of the centre O of the disc is constant and is v_0. Which of the following statements is false?

a) The angular velocity of the disc $\omega = \frac{v_0}{R}$.

b) The velocity of point A on the body is zero.

c) The velocity of point B is $v_B = \sqrt{2}v_0$.

d) The acceleration of point A on the body is zero.

e) Although there may be friction at contact point A, it does no work.

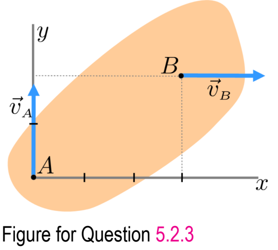


Figure for Question 5.2.3

Question 5.2.3. We know that points A and B on a rigid body (see Figure) moving in the x, y plane have, at a given instant, the following positions and velocities (in m and m/s):

$$\vec{r}_A = (0,0) \ ; \ \vec{v}_A = (0,3) \ ; \ \vec{r}_B = (3,2) \ ; \ \vec{v}_B = (v_B, 0)$$

With respect to the angular velocity ω of the body (in rad/s) and v_B (in m/s), which of the following answers is true?

a) $v_B = 3$ and $\omega = 2$

b) $v_B = 2$ and $\omega = 2$

c) $v_B = 2$ and $\omega = 1$

d) $v_B = \sqrt{13}$ and $\omega = 1$

e) Without prior knowledge of v_B, nothing can be deduced.

Question 5.2.4. A disc of mass m rolls down an inclined plane for a distance d without sliding. If the angle of the plane to the horizontal is α and the disc-plane friction coefficient is μ, the energy dissipated by the frictional force is given by:

a) 0

b) $d\mu mg \cos\alpha$

c) $d\mu mg \sin\alpha$

d) $d\mu mg \tan\alpha$

e) $d\mu mg$

Question 5.2.5. An equilateral triangle of total mass m is made up of three rods of length b and is arranged as shown in the figure. Its moment of inertia about the z-axis is

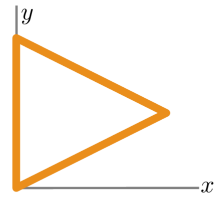


Figure for Question 5.2.5

a) $\frac{3mb^2}{7}$

b) $\frac{mb^2}{3}$

c) $\frac{mb^2}{2}$

d) $\frac{3mb^2}{4}$

e) $\frac{mb^2}{4}$

Question 5.2.6. A ring of radius r and mass m rolls down an inclined plane at an angle α without sliding. At a given instant, its centre of mass has a velocity v. At this instant, the modulus of the angular momentum of the ring with respect to point O (see Figure) is:

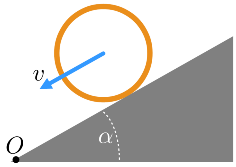


Figure for Question 5.2.6

a) mvr

b) $\frac{3mvr}{2}$

c) $2mvr$

d) $\frac{mvr}{2}$

e) $mvr \sin \alpha$

Question 5.2.7. A dancer who wants to increase her rotation speed has to bring her arms closer to her body because this:

a) increases the angular momentum.

b) reduces the effort.

c) increases the momentum.

d) reduces the moment of inertia.

e) increases the physical resistance.

Question 5.2.8. Four particles, $m_1 = m_3 = 3\,\text{kg}$ and $m_2 = m_4 = 4\,\text{kg}$ are at the vertices of a square, joined by rods of negligible mass. The length of the side of the square is $L = 2\,\text{m}$. The moment of inertia with respect to an axis perpendicular to the plane of the particles and passing through m_4 is:

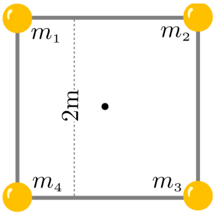


Figure for Question 5.2.8

a) $88\,\text{kg}\,\text{m}^2$

b) $40\,\text{kg}\,\text{m}^2$

c) $56\,\text{kg}\,\text{m}^2$

d) $20.5\,\text{kg}\,\text{m}^2$

e) $19.8\,\text{kg}\,\text{m}^2$

Question 5.2.9. A bar of length L is held horizontally at a height H above a table (see Figure). It is released and falls while maintaining its orientation. When it has descended a distance H, one of its ends touches one end of the table and the bar begins to rotate without friction as the end remains fixed to the table. The angular velocity at which the bar starts to rotate is:

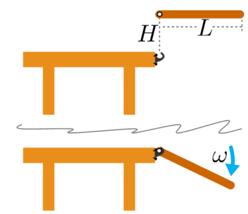


Figure for Question 5.2.9

a) $\frac{\sqrt{2gH}}{3L}$

b) $\frac{\sqrt{gH}}{L}$

c) $\frac{3\sqrt{2gH}}{2L}$

d) $\frac{\sqrt{2gH}}{L}$

e) $\frac{3\sqrt{gH}}{2L}$

Question 5.2.10. A weight lifter lifts weight m_2 in order to change it, keeping m_1 touching the ground. When the barbell forms an angle of 30°, it falls from his hands. What is the angular acceleration α of the assembly at this instant?

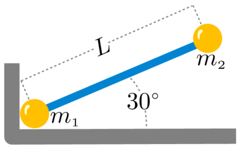


Figure for Question 5.2.10.

Data: The weights can be considered as point-like, the mass of the bar is negligible and $L = 2\,\text{m}$.

a) 2.5 rad/s²

b) 9.8 rad/s²

c) 4.2 rad/s²

d) 19.2 rad/s²

e) 4.9 rad/s²

Question 5.2.11. Which of the following statements is true?

a) If two bodies have the same dimensions and the same mass, their moment of inertia about the same axis is equal.

b) Steiner's theorem shows that the moment of inertia about an axis through the CM is less than it is about any other axis parallel to it.

c) Like mass, the moment of inertia is a characteristic quantity of the body.

d) The moment of inertia of a body about an axis depends on the angular velocity of the body.

e) Two bodies of different masses always have different moments of inertia about the same axis.

Question 5.2.12. The homogeneous cubic block in the figure slides without friction at a velocity v along the horizontal floor until it encounters a chock and becomes hooked to it, although this does not prevent it from rotating freely. The angular velocity of the block just after colliding with the chock is:

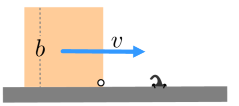


Figure for Question 5.2.12

a) $\frac{v}{2b}$

b) $\frac{5v}{7b}$

c) $\frac{9v}{7b}$

d) $\frac{v}{b}$

e) $\frac{3v}{4b}$

Question 5.2.13. A disc of radius $R = 25\,\text{cm}$ rotates about an axis of symmetry, fixed and frictionless, with an angular velocity of $\omega_0 = 37\,\text{rad/s}$. The moment of inertia about the axis is $I = 0.5\,\text{kg m}^2$. To stop it, we apply a force of $F = 2\,\text{N}$ to a brake pad of friction coefficient $\mu = 1.5$. The time it takes for the disc to stop rotating is:

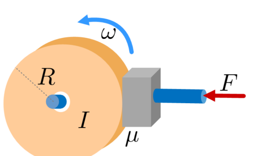


Figure for Question 5.2.13

a) $34.7\,\text{s}$

b) $54.7\,\text{s}$

c) $14.7\,\text{s}$

d) $44.7\,\text{s}$

e) $24.7\,\text{s}$

Question 5.2.14. A homogeneous bar of length $h = 0.8$ m and mass $m = 0.40$ kg can rotate without friction about a fixed point A. A point object of mass $m_b = 0.05$ kg and velocity $v_b = 10$ m/s impacts it horizontally and becomes embedded in the upper end of the bar. The modulus of the velocity of the lower end of the bar just after impact will be:

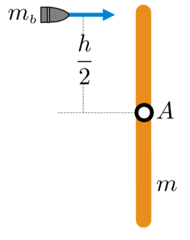


Figure for Question 5.2.14

a) 1.73 m/s

b) 3.73 m/s

c) $5.73\ \mathrm{m/s}$

d) $4.73\ \mathrm{m/s}$

e) $2.73\ \mathrm{m/s}$

Question 5.2.15. What is the value of the pulley's moment of inertia with respect to the axis normal to the plane in the figure passing through point A? The pulley consists of a ring of radius R and mass m, and a rod of length R and mass m.

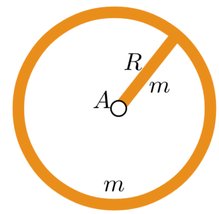


Figure for Question 5.2.15

a) $13/12\ mR^2$

b) $4/3\ mR^2$

c) $5/6\ mR^2$

d) $7/12\ mR^2$

e) mR^2

Question 5.2.16. As shown in the figure, the moment of inertia about the axis of a rigid cylinder of radius R, density ρ, and a cylindrical longitudinal hole of radius $R/2$ is:

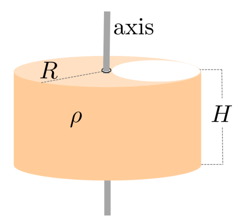


Figure for Question 5.2.16

a) $\frac{13}{50}\rho\pi H R^4$

b) $\frac{13}{21}\rho\pi H R^4$

c) $\frac{13}{21}\rho\pi H R^3$

d) $\frac{13}{32}\rho\pi R^4$

e) $\frac{13}{32}\rho\pi H R^4$

Problem 5.2.4. A small sphere of mass m and moving at a horizontal velocity of $20\,\mathrm{m/s}$ hits and remains attached to the end of a rigid piece hanging vertically from joint A, which consists of a bar and a disc, both homogeneous of mass m (see Figure). What is the angular velocity of the assembly just after the impact?

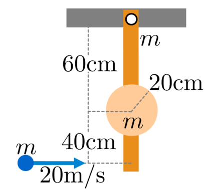


Figure for Problem 5.2.4

Solution

The moment of inertia with respect to point A after impact I_A is

$$I_A = \frac{1}{12}m1^2 + m0.5^2 + \frac{1}{2}m0.2^2 + m0.6^2 + m1^2 = 1.713333m$$

Using the conservation of angular momentum with respect to point A before and after impact, we have

$$m \cdot 20 \cdot 1 = I_A \omega$$

from which we obtain

$$\omega = 11.6732\,\text{rad/s}$$ ■

Problem 5.2.5. Determine the equation of motion of a ring of radius R rolling on an inclined plane without sliding.

Problem 5.2.6. During the Barcelona Olympics (1992), a large number of people from all over the world came to Barcelona. If we imagine the Earth as a homogeneous sphere of mass $M = 6 \times 10^{24}$ kg and a total of one million people, with an average mass of 80 kg, came to Barcelona:

a) Prove that the ratio between the periods before (T) and during the Olympics (T_{Ol}) is $\frac{T_{\text{Ol}}}{T} = 1 + \frac{5m}{3M}(\frac{3}{2}\cos^2\theta - 1)$, where m is the mass of all the people displaced and θ the latitude.

b) Calculate this relationship numerically for the case of the Barcelona Olympics.

Solution: b) $\frac{T_{\text{Ol}}}{T} \precsim 1$.

Problem 5.2.7. A homogeneous rigid sphere of mass m rolls without sliding along a horizontal plane accelerated by a force F parallel to the plane applied at CM. Calculate:

a) The acceleration of the sphere.

b) The dry frictional force of the plane.

Solution: a) $\frac{5F}{7m}$; b) $-\frac{2F}{7}$

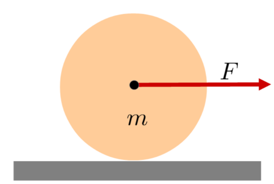


Figure for Problem 5.2.7

Problem 5.2.8. In the game of bowling, imagine we have a ball in a uniform sphere of 7 kg and 300 mm diameter, which we throw with an initial velocity of $v_0 = 6$ m/s and an initial angular velocity $\omega_0 = 0$. If the dynamic friction coefficient for ball and lane is $\mu = 0.1$, determine:

a) The time t_f it takes the ball to start rolling without sliding.

b) The velocity v_f and the angular velocity ω_f at the instant t_f.

Solution: a) $t_f = 1.747$ s; b) $v_f = 4.286$ m/s; $\omega_f = 28.57$ rad/s

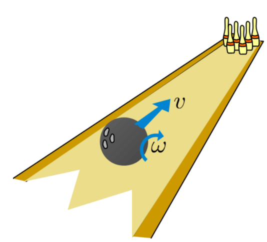


Figure for Problem 5.2.8

Problem 5.2.9. A flywheel mounted on a shaft is set in rotation by a body of mass m. The total moment of inertia about the axis shaft is I and M_f is the value of the moment of resistance forces of the shaft bearings.

a) What is the value of the wheel's angular acceleration?

b) What is the angular velocity when the body has descended a distance h?

Solution: a) $\alpha = \frac{mRg-M_f}{I+mR^2}$; b) $\omega = \sqrt{2\frac{mRg-M_f}{I+mR^2}\frac{h}{R}}$

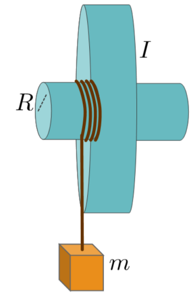


Figure for Problem 5.2.9

Problem 5.2.10. A uniform bar of 60 cm length and 15 N weight hangs from a frictionless pin. A bullet of mass 22.7 g impacts it at a velocity of 540 m/s and becomes embedded in it. Determine the angular velocity of the bar immediately after the impact.

Solution: $\omega = 29.3$ rad/s

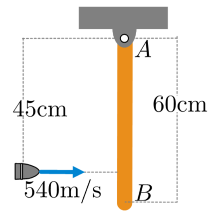


Figure for Problem 5.2.10

Problem 5.2.11. A homogeneous cylinder of radius $R = 0.4$ m rests unattached on the lorry bed at the cab end of an initially stopped lorry. The lorry starts up with a constant acceleration of $a = 0.7$ m/s².

If there is sufficient friction between the platform and the cylinder for it to roll without slipping, determine how long it would take for the cylinder to fall off the truck if the length of the platform is $L = 4.5$ m.

Solution: 4.19 s

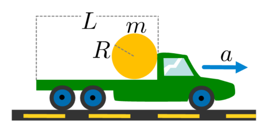


Figure for Problem 5.2.11

Question 5.3.1. The homogeneous bar in the figure can rotate freely around the fixed joint at the extreme left of the figure. If it starts from rest in a horizontal position, what will the modulus of its angular velocity be when it reaches the vertical position?

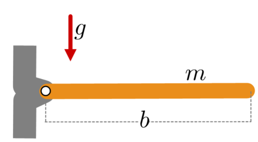


Figure for Question 5.3.1

a) $\sqrt{\frac{3g}{b}}$

b) $\sqrt{\frac{2g}{b}}$

c) $\sqrt{\frac{3g}{2b}}$

d) $\sqrt{\frac{3g}{5b}}$

e) $\sqrt{\frac{5g}{3b}}$

Question 5.3.2. An empty cylinder and a rigid cylinder of the same radius and mass are placed at the top of an inclined plane. They are released at the same instant and both roll without sliding. It is true that:

a) The two arrive at the base of the plane simultaneously, since they have the same mass, the same radius and they descend from the same height.

b) At each instant they both have the same kinetic energy.

c) Since there must be friction for them to roll, mechanical energy cannot be conserved in either cylinder.

d) At the base of the plane, both have the same kinetic energy.

e) Since the weight is applied at the CM, the angular momentum of each cylinder is conserved as they roll down the plane.

Question 5.3.3. A ring rolls without sliding. What is the quotient of its kinetic energy of rotation and its kinetic energy of translation, $\frac{E_{\text{rot}}}{E_{\text{trans}}}$?

a) 1

b) $\frac{1}{2}$

c) $\frac{1}{\sqrt{2}}$

d) 2

e) $\sqrt{2}$

Question 5.3.4. Indicate which of the following answers is true.

a) The change in kinetic energy of a rigid body is equal to the work of external forces.

b) The kinetic energy of a rigid body moving in a plane can be expressed as $E_c = \frac{1}{2}\, I_{CM}\, \omega^2$.

c) The variation in the kinetic energy of a system of particles is equal to the work of the internal forces.

d) The kinetic energy of a system of particles with total mass m can be expressed as $E_c = \frac{1}{2}\, m\, v_{CM}^2$.

e) The change in kinetic energy of a system of particles is equal to the work of external forces.

Question 5.3.5. Consider a four-wheel truck of mass $M + 4m$, where m is the mass of each wheel of radius R, and moment of inertia with respect to its axis $\frac{1}{8}mR^2$. The truck is moving in a straight line on a plane at speed v, without any wheels slipping. Thus, the truck's kinetic energy is:

a) $\frac{1}{2}(M + \frac{1}{8}m)v^2$

b) $\frac{1}{2}(3M + 7m)v^2$

c) $\frac{1}{3}(2M + 3m)v^2$

d) $\frac{1}{4}(2M + 9m)v^2$

e) $\frac{1}{4}(M + \frac{1}{8}m)v^2$

Question 5.3.6. The rigid bar in the figure hangs from joint A, without friction. A bullet hits the bar and becomes embedded at point B. The magnitude that is conserved during the impact is:

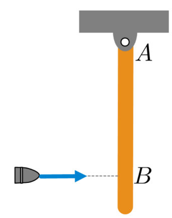


Figure for Question 5.3.6

a) The angular momentum about point B.

b) The mechanical energy.

c) The momentum.

d) The angular momentum about point A.

e) The horizontal component of the momentum.

Problem 5.3.6. A cylinder of mass M is attached by an inextensible and weightless rope to a body of mass m. For our purposes, the pulley has no mass and its axis has no friction. The cylinder rolls without sliding on the inclined plane, which has an angle of 45°.

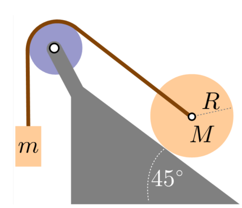


Figure for Problem 5.3.6

Data: $M = 20\,\text{kg}$, $m = 5\,\text{kg}$, $R = 0.3\,\text{m}$

Calculate:

a) The acceleration with which the system moves.

b) The tension of the rope.

Solution: a) $a = 2.56\,\text{m/s}^2$; b) $T = 61.85$ N

Problem 5.3.7. Consider the figure's fixed-axle, frictionless, homogeneous pulley, of mass M and moment of inertia I about the axis. It has two inextensible ropes of negligible mass, coiled without slipping. The first rope is coiled around radius r_1 with the other end fixed to a block of mass m_1, and the second, around radius r_2, with the other end fixed to a block of mass m_2.

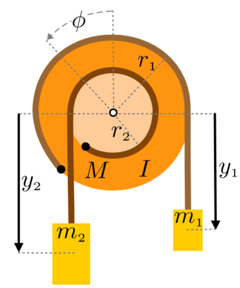


Figure for Problem 5.3.7

a) Write the mechanical energy of the system as a function of y_1, y_2, $\dot{y}_1$, $\dot{y}_2$, ϕ and $\dot{\phi}$.

b) Find the constraint relationships between $\dot{y}_1$, $\dot{y}_2$ and $\dot{\phi}$.

c) Find the energy of the system as a function of y_1, y_2, ϕ and $\dot{\phi}$ and the equation of motion for ϕ.

Using the numerical data: $r_1 = 40\,\text{cm}$; $r_2 = 30\,\text{cm}$; $m_1 = 15\,\text{kg}$; $m_2 = 40\,\text{kg}$; and $I = 4.6\,\text{kg m}^2$:

d) Find the angular acceleration of the pulley and the linear acceleration of the blocks. Indicate, for each block, whether the translation is upward or downward.

e) Starting from rest at the same level, $y_{10} = y_{20}$, what will be the distance between the blocks after $0.8\,\text{s}$?

Solution

a) The mechanical energy of the system as a function of y_1, y_2, $\dot{y}_1$, $\dot{y}_2$,ϕ and $\dot{\phi}$ is

$$E = \frac{1}{2}m_1\dot{y}_1^2 + \frac{1}{2}m_2\dot{y}_2^2 + \frac{1}{2}I\dot{\phi}^2 - m_1gy_1 - m_2gy_2$$

b) The constraints are $\dot{y}_1 = -r_1\dot{\phi}$ and $\dot{y}_2 = r_2\dot{\phi}$

c) The mechanical energy of the system as a function of y_1, y_2, ϕ and $\dot{\phi}$ is

$$E = \frac{1}{2}\left(m_1r_1^2 + m_2r_2^2 + I\right)\dot{\phi}^2 - m_1gy_1 - m_2gy_2$$

Applying the conservation of energy, we have

$$\dot{E} = 0 = \left(m_1r_1^2 + m_2r_2^2 + I\right)\dot{\phi}\ddot{\phi} + m_1gr_1\dot{\phi} - m_2gr_2\dot{\phi}$$

from which, isolating $\ddot{\phi}$, we find the equation of motion:

$$\ddot{\phi} = \frac{m_2r_2 - m_1r_1}{m_1r_1^2 + m_2r_2^2 + I}g$$

d) $\ddot{\phi} = 5.55283\,\text{rad/s}^2$; $\ddot{y}_1 = -r_1\ddot{\phi} = -2.22113\,\text{m/s}^2$; therefore, it goes up. $\ddot{y}_2 = r_2\ddot{\phi} = 1.66585\,\text{m/s}^2$; therefore, it goes down.

e) With the initial data at $t = 0$, $y_1(0) = y_{10}$, $y_2(0) = y_{20}$ and $\dot{y}_1(0) = 0$, $\dot{y}_2(0) = 0$, and taking into account that the accelerations are constant, we have

$$y_1 = y_{10} + \tfrac{1}{2}\ddot{y}_1 t^2$$
$$y_2 = y_{20} + \tfrac{1}{2}\ddot{y}_2 t^2$$

from which we obtain

$$y_2 - y_1 = \frac{1}{2}\left(\ddot{y}_2 - \ddot{y}_1\right) t^2$$

Substituting the accelerations and $t = 0.8\,\mathrm{s}$:

$$|y_2 - y_1| = 1.24383\,\mathrm{m}$$

■

Problem 5.3.8. Calculate the final CM velocity of a homogeneous sphere that is allowed to roll without sliding down an inclined plane to a drop h. Calculate it in the following ways.

a) Applying the equations of motion of the rigid body,

b) Applying the conservation of energy.

Solution: $v_{CM} = \sqrt{\frac{10}{7}gh}$

Problem 5.3.9. A homogeneous disc of radius 20 cm and mass 5 kg can rotate without friction in a vertical plane about its fixed axis. A rope of negligible mass is wound around it and a mass of 2 kg hangs from it and is dropped. The rope does not slide. Calculate the angular acceleration of the disc and the acceleration at which the 2 kg mass falls.

Solution: $\alpha = 21.8\ \mathrm{rad/s^2}$; $a = 4.36\ \mathrm{m/s^2}$

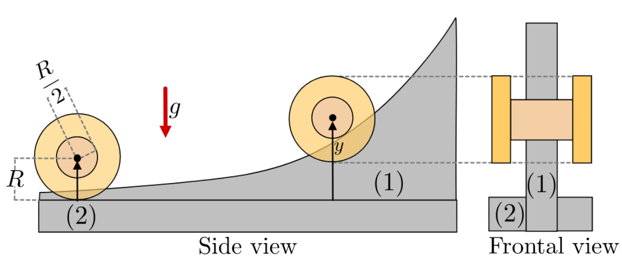


Figure for Problem 5.3.10

Problem 5.3.10. With an initial angular velocity ω_1, we drop (rolling without sliding) the homogeneous reel in the figure, of mass m and moment of inertia $I_{(CM)} = \frac{1}{3}mR^2$, down the guide (1) from a height $y = 3R$. At all times it rolls without sliding and there is no dissipative friction. What velocity of CM, v_2, and angular velocity ω_2, will the reel have after coming into contact with the horizontal plane (2)?

Note: Express the results as a function of ω_1, R and gravity g.

Solution: $v_2 = \sqrt{3gR + \frac{7}{16}R^2\omega_1^2}$; $\omega_2 = \sqrt{\frac{3g}{R} + \frac{7}{16}\omega_1^2}$

Problem 5.3.11. A small ball of mass $m = 100$ g bounces elastically and horizontally against the lower end, B, of a bar of mass $M = 6$ kg and length $L = 50$ cm, which is hinged without friction at the upper end A. The bar is initially at rest. If the velocity of the ball just before the collision is 30 m/s and it leaves in a horizontal direction, determine:

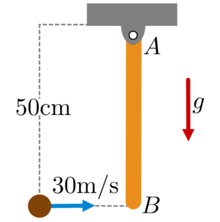


Figure for Problem 5.3.11

a) The modulus of the velocity of the ball and the angular velocity of the bar just after bouncing.

b) The maximum angle the bar reaches, with respect to the vertical (the initial angle is zero).

c) The modulus of the centre of mass velocity of the bar as it passes again through the initial position.

Solution: a) 27.143 m/s, 5.714 rad/s; b) 63.56°; c) 1.43 m/s

Problem 5.3.12. A Maxwell wheel is a device as shown in the figure. The rope has negligible mass; the axis has a radius r; the moment of inertia of the wheel and axle is I; and its mass is m. If starting from rest it descends a height h, how much are the final speeds of rotation and translation?

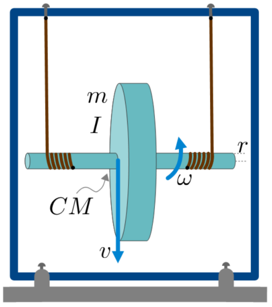


Figure for Problem 5.3.12

Solution: $v = \sqrt{\frac{2mgh}{m+I/r^2}}$ and $\omega = \sqrt{\frac{2mgh}{mr^2+I}}$

Problem 5.3.13. Find the equation of motion of the homogeneous bar of mass m and length L when it oscillates about an axis passing at a distance d from its centre of mass.

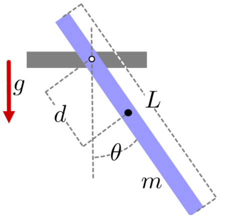


Figure for Problem 5.3.13

Solution: $\ddot{\theta} + \frac{gd}{\frac{1}{12}L^2+d^2}\sin\theta = 0$

→ 6P

6 Problems and questions

Question 6.2.1. A particle of mass m is dropped from a height h onto the pan of a scale, where it remains attached (see Figure). The pan has mass m and the balance is equivalent to a spring of recovery constant k. The period of the subsequent oscillations will be

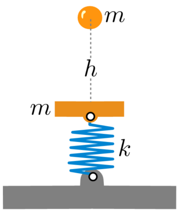


Figure for Question 6.2.1

a) $T = 2\pi\sqrt{\frac{m}{k}}$

b) $T = \pi\sqrt{\frac{8m}{k}}$

c) $T = 2\pi\sqrt{\frac{mh}{kg}}$

d) $T = 2\pi\sqrt{\frac{h}{g} + \frac{2m}{k}}$

e) $T = 2\pi\sqrt{\frac{m}{k} + \frac{h}{g}}$

Question 6.2.2. The three systems in the figure consist of springs of negligible mass with the same recovery constant and attached to the same body of mass m. There is no friction in any contact. In all cases, the body of mass m can perform a simple harmonic motion. If we designate T_1, T_2 and T_3 as the periods of oscillation in each case, it will be fulfilled that:

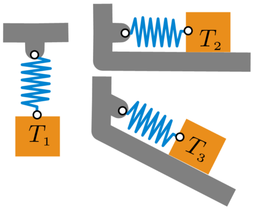


Figure for Question 6.2.2

a) $T_1 > T_2 = T_3$

b) $T_1 = T_2 < T_3$

c) $T_1 = T_2 = T_3$

d) $T_1 < T_2 = T_3$

e) It is necessary to know the value of the recovery constant to order the periods.

Question 6.2.3. A mass hanging from a spring oscillates as shown in the graph, which represents the elongation as a function of time. At time t_1, the mass has:

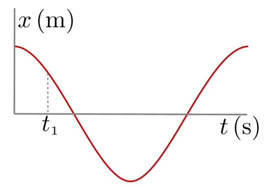


Figure for Question 6.2.3

a) positive velocity $\dot{x}$ and positive acceleration $\ddot{x}$.

b) positive velocity $\dot{x}$ and negative acceleration $\ddot{x}$.

c) negative velocity $\dot{x}$ and positive acceleration $\ddot{x}$.

d) negative velocity $\dot{x}$ and negative acceleration $\ddot{x}$.

e) positive velocity $\dot{x}$ and zero acceleration $\ddot{x}$.

Question 6.2.4. The mass $M = 10\,\text{kg}$ in the figure oscillates on a frictionless horizontal platform attached to two springs of recovery constants $k_1 = 100\,\text{N/m}$ and $k_2 = 50\,\text{N/m}$. Which of the following statements is true?

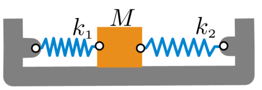


Figure for Question 6.2.4

a) The period of oscillation is 2.81 s.

b) The period of oscillation is 3.44 s.

c) The period of oscillation is 1.86 s.

d) If we tilt clockwise the platform by rotating it $30°$ with respect to the horizontal, the period of oscillation will be the same as when it is horizontal.

e) None of the other four answers is correct.

Question 6.2.5. When a simple pendulum oscillates:

a) The longer the length, the longer the period.

b) The shorter the length, the longer the period.

c) The longer the length, the shorter the period.

d) The period does not depend on the mass, unless the mass is small; thus, it increases with the mass.

e) All of the above answers are false

Problem 6.2.1. A homogeneous sphere of radius r and mass m rolls without sliding in a vertical plane on the inner surface of a fixed semi-cylinder of radius $R > r$. Find the mechanical energy and the period of the small oscillations as a function of R, r and g.

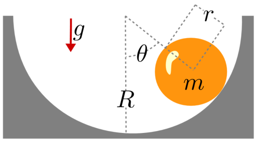


Figure for Problem 6.2.1

Solution

This is a conservative system with one degree of freedom:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 - mg\,(R-r)\cos\theta$$

with $I = \frac{2}{5}mr^2$ and $v = \omega r = (R-r)\,\dot{\theta}$. We obtain

$$E(\theta, \dot{\theta}) = \frac{7}{10}m\,(R-r)^2\,\dot{\theta}^2 - mg\,(R-r)\cos\theta$$

The equation of motion can be extracted from

$$\dot{E} = 0 = \frac{7}{5}m\,(R-r)^2\,\dot{\theta}\ddot{\theta} + mg\,(R-r)\sin\theta\;\dot{\theta}$$

which, for small oscillations, we can write as

$$\ddot{\theta} + \frac{5g}{7\,(R-r)}\theta = 0$$

Comparing this with the canonical expression of the SHM,

$$\omega_0 = \sqrt{\frac{5g}{7\,(R-r)}}$$

it corresponds to a period:

$$T_0 = 2\pi\sqrt{\frac{7\,(R-r)}{5g}}$$

■

Problem 6.2.2. A body has a simple harmonic motion with an amplitude of $5.2\,\mathrm{cm}$. When the elongation is $3.4\,\mathrm{cm}$, the velocity is $49.8\,\mathrm{cm/s}$. Do we have enough information to find the phase of the motion? How about for calculating the period? Find as much as you can from the information we have.

Solution: $T = 0.50\,\mathrm{s}$; $\theta = 0.7121\,\mathrm{rad}$ ($x = A\sin\theta$)

Problem 6.2.3. A $2\,\mathrm{kg}$ body is at rest on a smooth horizontal plane and is subjected to two horizontal springs of recovery constants $k_1 = 100\,\mathrm{N/m}$ and $k_2 = 200\,\mathrm{N/m}$. The length of each of the two non-deformed springs is $40\,\mathrm{cm}$. The free ends of the springs are stretched and attached to two fixed walls $120\,\mathrm{cm}$ apart. Determine the equilibrium position of the body. What is the frequency of oscillation about the equilibrium position?

Solution: $66.7\,\mathrm{cm}$; $f = 1.95\,\mathrm{Hz}$

Problem 6.2.4. A particle moving in simple harmonic motion has a velocity of $16\,\text{cm/s}$ and $12\,\text{cm/s}$ when it passes, respectively, $3\,\text{cm}$ and $4\,\text{cm}$ from the centre of vibration. Calculate its amplitude and period.

Solution: $A = 5\,\text{cm}$; $T = 1.57\,\text{s}$

Problem 6.2.5. A point mass performs a simple harmonic motion. When the elongation is $+10\,\text{cm}$ moving towards the equilibrium point, its kinetic energy is $10^{-5}\,\text{J}$ and its potential energy is also $10^{-5}\,\text{J}$. If the mass is $2\,\text{g}$, find the amplitude, period and phase of the motion.

Solution: $A = 14.1\,\text{cm}$; $T = 6.28\,\text{s}$; $\theta = 3\pi/4\,\text{rad}$ ($x = A\sin\theta$)

Problem 6.2.6. A cylindrical buoy of height $4\,\text{m}$, radius $2\,\text{m}$ and mass $40000\,\text{kg}$ floats vertically in water. If it makes small vertical oscillations, find its period of oscillation.

Solution: $T = 3.58\,\text{s}$

Problem 6.2.7. Two identical cylinders of mass M and cross-section S are arranged as shown in the figure and are partially immersed in water. Neglecting the mass of the pulley, the frictions and the inertia of the water, determine the period of oscillation of the system of weights when they are slightly separated from their equilibrium position.

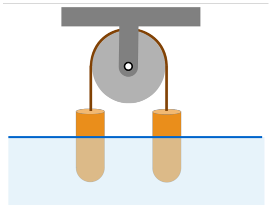


Figure for Problem 6.2.7

Solution: $T = 2\pi\sqrt{\frac{M}{Sg\rho}}$

Problem 6.2.8. In a U-shaped tube of constant cross-section, with both branches in a vertical position, a liquid occupying length L of the tube is introduced. Initially, it becomes unbalanced and consequently starts to oscillate around its equilibrium position. Assuming that the liquid is incompressible and that there is no friction, show that the liquid will oscillate with harmonic motion and determine the corresponding period.

Solution: $T = 2\pi\sqrt{\frac{L}{2g}}$

Problem 6.2.9. A $100\,\text{g}$ body hangs from a long spring. If we stretch it by lowering it $10\,\text{cm}$ below its equilibrium position and then release it, it vibrates with a period of $2\,\text{s}$.

a) With what speed does it pass through its equilibrium position?

b) What is its acceleration when it is $5\,\text{cm}$ above this position?

c) In the upward motion, how long does it take to move from a point 5 cm below its equilibrium position to a point 5 cm above it?

d) How much will the spring shorten when the body is unhooked?

Solution: a) direction ↑ and modulus $v = 31.4\,\mathrm{cm/s}$; b) direction ↓ and modulus $a = 49.3\,\mathrm{cm/s^2}$; c) $t = 0.33\,\mathrm{s}$; d) $\Delta\ell = 99.3\,\mathrm{cm}$

Problem 6.2.10. A 12 kg body hangs from a spring. If we stretch it until its length increases by 10 cm and release it, an oscillatory motion of period 1.45 s starts. Answer the following.

a) When oscillating and moving downwards, how long does it take the body to move from a point 3 cm above its equilibrium position to a point 6 cm below its equilibrium position?

b) What is the velocity of the body passing through this latter position?

c) Once at rest, how much will the spring shorten if we unhook the 12 kg body?

Solution: a) $t = 0.22\,\mathrm{s}$; b) $v = -0.347\,\mathrm{m/s}$; c) $\Delta\ell = 0.52$ m

Problem 6.2.11. Two springs of recovery constants $k_1 = 1200\,\mathrm{N/m}$ and $k_2 = 600\,\mathrm{N/m}$ are joined in series. The free end of k_1 hangs from a point and a body of mass $m = 10\,\mathrm{kg}$ hangs from the free end of k_2. Find:

a) The period of the free oscillations that the body can make.

b) Calculate this while also assuming that the springs are connected in parallel.

Solution: a) $T = 0.99\,\mathrm{s}$; b) $T = 0.47\,\mathrm{s}$

Problem 6.2.12. An inextensible wire of negligible mass passes through the throat of a pulley whose mass is concentrated at its periphery. A mass M hangs from one end and the other is attached to a vertical spring fixed to the ground (see Figure). If the mass of the pulley is $m = 800$ g, the mass of the hanging body is $M = 200$ g and the spring has a negligible mass and a recovery constant $k = 16\,\mathrm{N/m}$, calculate the period of the small oscillations in the system.

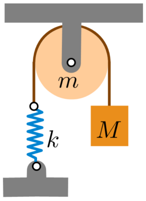


Figure for Problem 6.2.12

Solution: 1.57 s

Problem 6.2.13. By hanging a mass M from a spring (which we assume to be massless and initially non-deformed), it elongates by 2.5 m. In this situation, we push it upward at a velocity of $v = 2\,\mathrm{m/s}$. Find the trajectory of the mass M.

Solution: $y = 1.01\sin(1.98t)$ (SI)

Problem 6.2.14. A 1 kg body attached to the end of a spring starts its motion when at position $x = 1\,\mathrm{m}$ with an initial velocity of $v = 2\,\mathrm{m/s}$. If the period (T) of the movement is π s, calculate the maximum elongation of the spring (A) and find its trajectory. Calculate the maximum velocity and acceleration and the positions at which they occur.

Solution: (SI) $A = \sqrt{2}$; $x = \sqrt{2}\sin(2t + \pi/4)$; $v_{\text{max}} = \pm 2\sqrt{2}$ $(x = 0)$; $a_{\text{max}} = \pm 4\sqrt{2}$;$(x = \pm\sqrt{2})$

Problem 6.2.15. A simple model currently used to describe the proteins present in our body consists simply of balls representing the amino acids (of approximately 10^{-24} kg), held together by springs of recovery constant $k = 5 \times 10^{-20}\,\mathrm{N/m}$. Calculate the vibration frequency of the amino acids if we assume that this model is valid.

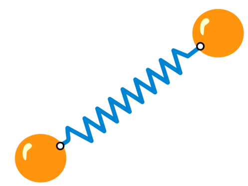


Figure for Problem 6.2.15

Solution: $f = 71\,\mathrm{Hz}$

Problem 6.2.16. We build a pendulum from two identical uniform rods a and b, each of length L and mass m, joined at right angles in the form of a **T**, with the centre of rod a joined to the end of b. We hang the pendulum from the free end of rod b, swinging it in a vertical plane.

a) Calculate the moment of inertia with respect to the axis of rotation.

b) Find the expressions for the kinetic and potential energy as a function of the angle of the axis of the pendulum with respect to the vertical.

c) Derive the equation of motion.

d) Find the period for the small oscillations.

Solution: a) $I = \frac{17}{12}mL^2$; b) $E_c = \frac{17}{24}mL^2\dot{\theta}^2$, $U = \frac{3mgL}{2}(1 - \cos\theta)$; c) $\ddot{\theta} + \frac{18}{17}\frac{g}{L}\sin\theta = 0$; d) $T = 2\pi\sqrt{\frac{17L}{18g}}$

Problem 6.2.17. The motion of a simple harmonic oscillator is described by the equation $x\,(t) = 4\sin\,(0.2t + 0.3)$, with x in m and t in s.

a) Calculate the amplitude, period, frequency and initial phase of the motion.

b) Determine the velocity and acceleration as a function of time, as well as the initial conditions.

c) What is the phase difference between the elongation and the velocity? Between the elongation and the acceleration?

d) Calculate the position, velocity and acceleration at $t = 5$ s.

Solution: a) $A = 4$ m; $\varphi_0 = 0.3$ rad; $f_0 = 0.032$ s^{-1}; b) $v(t) = 0.8\cos(0.2t + 0.3)$; $a(t) = -0.16\sin(0.2t + 0.3)$ (SI units); $x(0) = 1.18$ m; $v(0) = 0.76$ $^{\text{m}}/_{\text{s}}$; c) $^{\pi}/_{2}$; π; d) $x(5) = 3.85$ m ;$v(5) = 0.21$ $^{\text{m}}/_{\text{s}}$; $a(5) = -0.154$ $^{\text{m}}/_{\text{s}^2}$

Problem 6.2.18. A 1 kg particle performs a simple harmonic motion with an amplitude of 0.5 m. At instant $t = 0$, it passes through the equilibrium position with a velocity of $\dot{x}(0) = +2\,\text{m/s}$.

a) Calculate the frequency and period.

b) Determine the elongation and velocity as a function of time.

c) Calculate the force and the kinetic and potential energies when the particle is at 0.2 m from its equilibrium position.

Solution: a)$f_0 = 0.637$ Hz; $T_0 = 1.57$ s; b) $x = 0.5\sin(4t)$; $v = 2\cos(4t)$; c) $F = 3.2$ N; $E_c = 1.68$ J; $U = 0.32$ J

Problem 6.2.19. In the mechanism shown in the figure, the spring recovery constant is $k = 100\,\text{N/m}$; the mass of the homogeneous cylindrical pulley is $M = 4\,\text{kg}$; and the radius $R = 30\,\text{cm}$. The mass of the block is $m = 1\,\text{kg}$. The rope does not slide at any time and there is no friction on the axis. Find the equation of motion and the period.

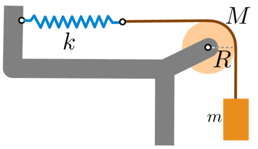


Figure for Problem 6.2.19

Solution: $\ddot{x} + 33.33\,x = 0$ (SI units); 1.088 s

Problem 6.2.20. The pulley, the springs (of recovery constants $3k$ and k) and the inextensible rope in the figure all have a negligible mass. The bar, of mass m, moves slightly and vertically from the horizontal equilibrium position. What is the period of the oscillations?

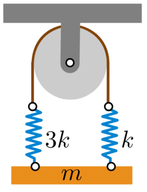


Figure for Problem 6.2.20

Solution: $2\pi\sqrt{\frac{m}{3k}}$

Question 6.3.1. The vibration amplitude of a damped oscillator decreases from 75 mm to 70 mm in one cycle. The oscillating mass is 1.2 kg and the time it takes to go from the centre to the end of the oscillation is 0.5 s. The damping constant of the viscous friction force is:

a) $82.8 \times 10^{-3}\,\text{N s/m}$

b) $165.6 \times 10^{-3}\,\text{N s/m}$

c) $34.5 \times 10^{-3}\,\text{N s/m}$

d) $331.2 \times 10^{-3}\,\mathrm{N\,s/m}$

e) $69.0 \times 10^{-3}\,\mathrm{N\,s/m}$

Question 6.3.2. With no friction from the ground, a mass M of 100 g can oscillate horizontally relative to its equilibrium position under the action of four springs of recovery constants $k_1 = 10\,\mathrm{N/m}$, $k_2 = 20\,\mathrm{N/m}$, $k_3 = 10\,\mathrm{N/m}$ and $k_4 = 15\,\mathrm{N/m}$, together with two devices of damping constants $b_1 = 1/2\,\mathrm{N\,s/m}$and $b_2 = 1\,\mathrm{N\,s/m}$. We can state that:

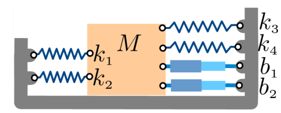


Figure for Question 6.3.2

a) In SI units, the equation of motion is $\ddot{x} + 15\dot{x} + \frac{380}{3}x = 0$

b) In SI units, the equation of motion is $\ddot{x} + \frac{10}{3}\dot{x} + 550x = 0$

c) In SI units, the equation of motion is $\ddot{x} + \frac{10}{3}\dot{x} + \frac{600}{19}x = 0$

d) The equivalent damping constant is $1/3\,\mathrm{N\,s/m}$.

e) The period of the oscillations will be 0.283 s.

Question 6.3.3. A system that can oscillate has critical damping. In this case:

a) The pulsation is small.

b) All oscillations have the same duration.

c) The motion is aperiodic.

d) The elongation is a positive exponential function of time.

e) None of the above answers is valid.

Problem 6.3.2. A damped oscillator has a mass of 50 g and a period of oscillation of 2 s. Its amplitude decreases by 5% every cycle. Assuming the equation of motion in the form $m\,\ddot{x} + b\,\dot{x} + k\,x = 0$, find b and k.

Solution: $b = 2.56\,\mathrm{g/s}$; $k = 0.49\,\mathrm{N/m}$

Problem 6.3.3. A 0.5 g particle executes an oscillatory motion with the damping being proportional to the velocity. The frequency of vibration is 0.5 Hz and the amplitude is halved after 10 s. Calculate:

a) The damping factor.

b) If the movement were caused by a spring, the recovery constant k of the spring.

c) The frequency it would have if there were no damping.

Solution: a) $\gamma = 0.069\ \mathrm{s}^{-1}$; b) $k = 4.94{\cdot}10 - 3\,\mathrm{N/m}$; c) $f = 0.5\,\mathrm{Hz}$

Problem 6.3.4. A mass m is hung from a vertically suspended spring with an elongation of $9.8\,\mathrm{cm}$. The mass is stretched downwards and released, resulting in oscillations. Determine what value the coefficient b must have in order for:

a) The oscillations to end after $100\,\mathrm{s}$ (consider that they have ended when the amplitude has been reduced to one-thousandth of the initial value).

b) The period of the oscillations to be twice that of the natural period.

c) The mass to periodically return to the equilibrium position.

Solution: (in SI units): a) b = $0.138\,\mathrm{m}$; b) $b = 17.3\,\mathrm{m}$; c) $b \geq 20\,\mathrm{m}$

Problem 6.3.5. A body hangs from a spring capable of oscillating vertically. When there is no damping the period is $2\,\mathrm{s}$; when the damping is switched on, it has a period of $2.7\,\mathrm{s}$.

a) Determine the equation for the elongation assuming that it is $10\,\mathrm{cm}$ at the initial instant and the velocity is zero.

b) Find the factor a by which the damping constant must be multiplied in order to reach the critical value.

Solution: a) $x(t) = 13.5\ \mathrm{e}^{-2.11t}\,\sin(2.33t + 0.83)$; b) $a = 1.49$

Problem 6.3.6. A simple pendulum with a mass of $10\,\mathrm{g}$ has a period of $2\,\mathrm{s}$ and an amplitude of 2°. Due to friction with the air, a resisting force $-bv$ that is proportional to the velocity acts on the pendulum. Calculate the damping constant b, knowing that the amplitude is reduced to 1.3° after ten complete oscillations.

Solution: $b = 4.31 \times 10^{-4}\,\mathrm{N\,s\,m^{-1}}$

Problem 6.3.7. A pendulum consists of a thread of 1 m length and negligible mass, from which hangs a rigid sphere of 1 cm radius and 8 g mass. The pendulum moves immersed in a liquid of density $1\,\mathrm{g/cm^3}$. The sphere is subjected to its weight, the Archimedean thrust, the tension of the thread and a viscous force $-bv$, where $b = 1.6 \times 10^{-4}\,\mathrm{N\,s/m}$. Find:

a) The period of the small oscillations assuming that $b = 0$.

b) The period of the small oscillations with the value of b in the statement.

c) The time it takes for the amplitude of the movement to go from $\pi/36$ rad to $\pi/72$ rad.

d) The value of b for the damping to be critical.

Solution: a) $T_0 = 2.91\,\text{s}$; b) $T = 2.91\,\text{s}$; c) $\Delta t = 69.31\,\text{s}$; d) $b = 0.034\,\text{N}\,\text{s}\,\text{m}^{-1}$

Problem 6.3.8. A mass is subjected to a force $\vec{F} = -k\vec{x}$ and a viscous force $\vec{F}_v = -b\vec{v}$ with $b = 5\,\text{Ns}\,\text{m}^{-1}$. The elongation is given by $x(t) = 0.8\,\text{e}^{-2.5t}\sin(\pi t - 0.3)$, with x in m and t in s.

a) Calculate the mass and the elastic recovery constant.

b) Calculate the position and velocity when $t = 1.2\,\text{s}$.

c) Determine the times at which the mass passes through its equilibrium position with positive and negative velocity.

d) Determine the times when the mass is at the extremes with positive and negative x.

Solution: a) $m = 1\,\text{kg}$; $k = 16.1\,\frac{\text{N}}{\text{m}}$; b) $x(1.2\,\text{s}) = 1.28\,\text{cm}$; $v(1.2\,\text{s}) = -0.0862\,\text{m}\,\text{s}^{-1}$; c) $t = \frac{0.3}{\pi} \pm N$; d) $t = \frac{1}{\pi}(\arctan\frac{\pi}{2.5} + 0.3) \pm N = 0.81538 \pm N$

Problem 6.3.9. A 2 kg particle is subjected to an elastic force of recovery constant $k = 10\,\text{N}\,\text{m}^{-1}$. If we introduce the oscillating system into a viscous medium, the period increases by 10%.

a) Determine the expressions for elongation, velocity and acceleration as a function of time, taking the origin of time when $x > 0$ and $v = 0$ and knowing that the elongation at this instant is 5 m.

b) Calculate the value of the amplitude after one cycle and its relation to the initial amplitude.

c) How much should the damping constant be for it not to oscillate?

Solution: a)

$$\begin{aligned}
x(t) &= 5.5\,\text{e}^{-0.935t}\sin(2.03t + 1.14)\\
v(t) &= \tfrac{dx}{dt} = 11.2\,\text{e}^{-0.935t}\cos(2.03t + 1.14) - 5.14\,\text{e}^{-0.935t}\sin(2.03t + 1.14)\\
a(t) &= \tfrac{dv}{dt} = -17.9\,\text{e}^{-0.935t}\sin(2.03t + 1.14) - 20.9\,\text{e}^{-0.935t}\cos(2.03t + 1.14)
\end{aligned}$$

b) $A(T) = 0.304\,\text{m}$; 0.055; c) $b \geq 4\sqrt{5}\,\text{kg}\,\text{s}^{-1}$

Question 6.4.1. The equation of motion of a forced oscillator is $\ddot{x} = -\frac{7}{4}\dot{x} - 9x + \sin 4t$. It is true that, in steady state:

a) The system is in velocity resonance.

b) The amplitude of the elongation decreases exponentially.

c) The elongation has the form $x = A_p \sin(4t - \frac{3\pi}{4})$.

d) The velocity has the form $\dot{x} = v_0 \sin(4t + \frac{\pi}{2})$.

e) The elongation has the form $x = A_p \cos(3t + \frac{\pi}{4})$.

Question 6.4.2. After five oscillations, the amplitude of an oscillator is smaller by a factor of e^{-1}. If ω_0 is its natural pulsation, the amplitude resonance pulsation will be:

a) $\Omega_{RA} = 0.999\omega_0$

b) $\Omega_{RA} = 0.5\omega_0$

c) $\Omega_{RA} = 1.125\omega_0$

d) $\Omega_{RA} = 0.2\omega_0$

e) $\Omega_{RA} = \frac{1}{3}\omega_0$

Question 6.4.3. Once the steady state is reached for a forced oscillator with a damping force proportional to the velocity, being b the damping constant, it is true that:

a) At elongation resonance, the mechanical impedance is b.

b) At velocity resonance, the average power dissipated by the friction force in one cycle is minimal.

c) At elongation resonance, the phase difference between the applied harmonic force and the elongation is zero.

d) In velocity resonance, the phase difference between the applied harmonic force and the elongation is $\frac{\pi}{2}$ rad.

e) The frequency of velocity resonance is smaller than that of elongation resonance.

Question 6.4.4. A particle of mass $0.2\,\mathrm{kg}$ is given an elastic force of recovery constant k and a velocity-proportional damping force of constant $4\,\mathrm{N\,s/m}$. A force $F = F_0 \cos \omega_1 t$ is applied to the particle and, after reaching steady state, the particle oscillates with elongation $x = 0.6 \cos(\omega_1 t - \frac{\pi}{2})$ (all in SI units). We can state that:

a) If we increase the frequency of the force F, the amplitude of the particle's elongation x will increase.

b) The mechanical impedance is $4\,\mathrm{N\,s/m}$.

c) There is a lack of data to calculate the mechanical impedance.

d) If we increase the frequency of the force F, the amplitude of the particle velocity will increase.

e) The maximum velocity of the particle must be $0.6\,\mathrm{m/s}$.

Question 6.4.5. A particle with a mass of $2\,\mathrm{kg}$ performs forced oscillations in a viscous medium under the action of an external force $F = 5\cos(3t)$ (all in SI units). In steady state, we can state the following:

a) The elongation could be $x = A\cos(3t + \frac{\pi}{4})$.

b) The acceleration of the particle could be $a = 2.5\cos(3t)$.

c) The velocity of the particle could be $v = v_0 \cos(2t)$.

d) If the acceleration were $a = -0.2\cos(3t - \frac{\pi}{2})$, the particle would be in velocity resonance.

e) If the velocity were $v = 0.5\sin(3t)$, the particle would be in velocity resonance.

Question 6.4.6. If a harmonic force $F(t)$ is applied to a damped harmonic oscillator and the amplitude of the oscillations it makes is maximum, the frequency of the force is:

a) Slightly greater than the system's own frequency of vibration in the absence of damping.

b) Greater than the system's frequency of vibration in the absence of $F(t)$ when damped.

c) Smaller than the system's frequency of vibration in the absence of $F(t)$ when damped.

d) Equal to the frequency at which the system would oscillate when damped and without applied force.

e) Equal to the free vibration frequency of the system.

Question 6.4.7. Which of the following statements is true?

a) In a mass-spring-damper system excited by a harmonic force, the amplitude and velocity resonance pulsations coincide if the viscous force is negligible.

b) The mechanical impedance of a mass-spring-damper system is equal to the coefficient of proportionality between the viscous force and the velocity.

c) The average power dissipated by forced harmonic oscillations is maximum when there is amplitude resonance.

d) The period of the damped harmonic oscillations depends on the amplitude.

e) None of the other four statements is correct.

Question 6.4.8. In damped motion, where x is the elongation and $F_f = -b\dot{x}$ is the frictional force, which of the following statements is true?

a) If it is oscillatory, the period is smaller than that of an undamped system.

b) If the damping factor γ were equal to the pulsation of the system without damping, the motion would not be oscillatory.

c) If it is oscillatory, the smaller the damping factor γ is, the faster the amplitude decreases.

d) None of the other four statements is true.

e) If the system were overdamped and, in addition, we were to apply an external harmonic force so that it would be in elongation resonance, under these conditions it would also oscillate with the same pulsation as that one of the overdamped motion.

Question 6.4.9. A mass of $1\,\text{kg}$ attached to a damper and a spring of respective recovery constants $2\,\text{N s/m}$ and $5\,\text{N/m}$ is brought into resonance by an external agent applying a force $F_0 \sin(\Omega t)$. We can state that:

a) If $\Omega = 2\,\text{rad/s}$, the system is in velocity resonance.

b) If $\Omega < 1\,\text{rad/s}$, the system does not oscillate.

c) The mechanical impedance of the system is $2\,\text{N s/m}$ when we have amplitude resonance.

d) The elongation is out of phase by $\frac{\pi}{3}$ rad with respect to the force applied by the external agent when we have amplitude resonance.

e) The elongation is in phase with the force applied by the external agent when we have amplitude resonance.

Problem 6.4.1. A $1\,\text{kg}$ mass is attached to a structure by an elastic spring and subjected to a force $F = F_0 \sin(\Omega t)$, with $F_0 = 2.5\,\text{N}$ and Ω variable. Using the observed relationship indicated in the table between Ω and the amplitude A_P, estimate the spring recovery constant k and the damping constant b.

Table for Problem 6.4.1

$\Omega\ (\text{s}^{-1})$	14	20	26	32	36	40
A_p(cm)	0.31	0.42	0.78	1.10	0.85	0.41

Solution: $k = 1089\,\text{N m}^{-1}$; $b = 6.9\,\text{N s m}^{-1}$

Problem 6.4.2. A mass of $3\,\text{g}$ is subjected to a restoring force of $1\,\text{N/m}$ and a damping force of $0.1\,\text{N s/m}$. If a force $F = 0.1\ \cos(10\pi t)$ (SI units) is applied, calculate the amplitude and the phase difference between the force and the velocity. Also calculate the mechanical impedance and find the velocity resonance frequency.

Solution: $A_P = 2.70\,\text{cm}$; $\theta = 32.0°$; $Z = 0.12\,\text{N s m}^{-1}$; $f_{RV} = 2.91\,\text{Hz}$

Problem 6.4.3. A $10\,\text{g}$ particle is subjected to the action of a restoring force of $0.05\,\text{N/m}$ and a damping force of $0.03\,\text{Ns/m}$. If a periodic force of $50\,\text{rad/s}$ of pulsation and $0.001\,\text{N}$ of amplitude acts on this particle, find, in steady state:

a) The mechanical impedance.

b) The maximum velocity.

c) The velocity resonance frequency.

d) The velocity amplitude in this case.

Solution: a) $Z = 0.5\,\text{N s m}^{-1}$; b) $v_{\text{max}} = 0.20\,\text{cm s}^{-1}$; c) $f_{RV} = 0.36\,\text{Hz}$; d) $v_{\text{max}} = 3.33\,\text{cm s}^{-1}$

Problem 6.4.4. A body of mass 5 kg is hanging from the end of a spring. It is separated from its equilibrium position and is observed to perform a vertical SHM, which takes 0.4 s to go from one end of the oscillation to the other. The mechanical energy of the particle is 100 J.

a) Calculate the time taken by the body to go in a downward motion from the position 0.5 m above the centre of oscillation to 0.2 m below the centre.

If we oscillate the above system in a viscous medium, the frequency becomes 90% of what it was in the SHM.

b) Calculate the amplitude reduction factor over a 0.5 s interval.

We apply to the system a harmonic force of the same frequency as that of the damped oscillation. It is observed that, in steady state, the system reaches a maximum velocity of $1.5\,\mathrm{m\,s^{-1}}$.

c) What is the amplitude of the applied force?

Solution: a) $t = 0.117$ s; b) $\frac{A}{A_0} = 0.181$; c) $F_0 = 52.8$ N

Problem 6.4.5. A mass of 3 kg undergoes a simple harmonic motion in the direction of the x-axis with amplitude 10 cm and a period of 3 s. For $t = 2.5$ s, the mass passes through the equilibrium position, $x = 0$, with positive velocity.

a) Determine the elongation and velocity at $t = 0$.

At an instant when the mass passes through the equilibrium position while moving in the positive direction of the x-axis, a damping device is set in motion which provides a viscous friction force proportional to the velocity and with a coefficient of $10\,\mathrm{N\,s/m}$.

b) Taking as a new time origin ($t = 0$) the instant when the damper is started, write the expression of the trajectory and determine all the parameters involved.

c) Calculate the kinetic energy of the system at the end of the first oscillation cycle.

Finally, a periodic force $F = 5\cos(\Omega t)$ (SI units) is applied to the mass, also in the direction of the x-axis.

d) What would the amplitude of the forced oscillations be if the system were in velocity resonance?

Solution: a) $x_0 = 0.087$ m, $v_0 = 0.105$ m/s
b) $x = 0.165\ \ \mathrm{e}^{-1.67t}\cos(1.27t + 3\pi/2)$; c) $E_c = 4.4 \times 10^{-9}$ J; d) $A = 0.239$ m

Problem 6.4.6. A block of mass $m = 3\,\text{kg}$ on a frictionless horizontal plane is attached to two springs of recovery constants $k_1 = 7\,\text{N/m}$ and k_2 , and to a damper of constant $b = 10\,\text{N s/m}$. A harmonic force $F = 4\sin(2t)$, in SI units, is applied to the block.

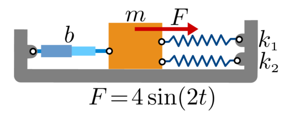


Figure for Problem 6.4.6

a) Determine the value of k_2 so that in the oscillatory motion of the block the phase difference between the harmonic force and the elongation is $\frac{\pi}{2}$ rad.

We eliminate the spring of recovery constant k_2.

b) Write the equation of the elongation as a function of time and determine all the involved parameters.

c) What is the maximum speed reached by the block in its oscillation?

When the block is at the right end of the oscillation we eliminate the harmonic force.

d) What kind of damped motion will the block make?

e) Determine the equation that gives the position of the block as a function of time and calculate all the involved parameters.

Solution: a) $k_2 = 5$ N/m; b) $x = 0.194 \quad \sin(2t - 1.82)$; c) $v_{\text{max}} = 0.388\,\text{m s}^{-1}$; d) the system is overdamped; e) $x = 0.340 \quad e^{-t} - 0.146 \ \ e^{-2.33t}$

Problem 6.4.7. A **U**-shaped tube with a cross-section of 1.8 cm in diameter contains 120 g of ethanol ($\rho = 787.4\,\text{kg/m}^3$). In one of the branches, a small displacement of the liquid is provoked.

a) If there is no friction between the liquid and the walls of the tube, find the natural pulsation of its oscillations.

When these oscillations are observed, it is found that their amplitude decreases by 3.5% in each period.

b) What is the period of the damped oscillations?

c) With what frequency should the liquid in the tube be blown so that the oscillating movement has the maximum amplitude?

Solution: a) $5.72\,\text{rad s}^{-1}$; b) 1.1 s; c) 0.91 Hz

Problem 6.4.8. A mass m_1 has been added to a disc D of mass m_D, which is attached to a spring (of recovery constant k and equivalent mass m_u) and, through a rope and a pulley P_2, to a second mass m_2. The disk causes an aerodynamic friction of coefficient b. The system can be forced to oscillate by an engine that moves the spring harmonically with a pulsation Ω. The pulleys P_1 and P_2 have negligible mass.

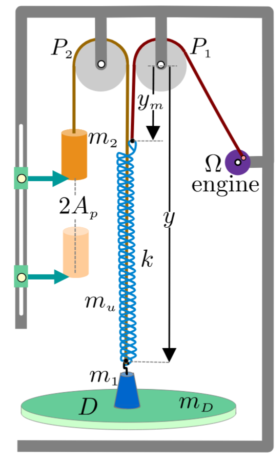


Figure for Problem 6.4.8

With the engine stopped, we move the mass m_2 vertically and release it:

a) Write the equation of motion using the y-coordinate.

We start the engine so that $y_m = y_{mo} + R\sin(\Omega t + \theta_0)$.

b) Find the new equation of motion.

c) Write the equation of motion in canonical form (call x the new coordinate) and identify all the parameters involved.

d) Plot the amplitude of the forced stationary oscillations as a function of engine pulsation.

Data: $m = m_1 + m_D + m_2 + m_u = 0.80\,\mathrm{kg}$; $k = 9.7\,\mathrm{N/m}$; $R = 4\,\mathrm{cm}$; $b = 1.5\,\mathrm{N\,s/m}$

Solution (in SI units):

a) $m\ddot{y} + b\dot{y} + k(y - y_m - \ell_{eq}) = 0$, with $y_{eq} - y_m - \ell_{eq} = 0$

b) $m\ddot{y} + b\dot{y} + k(y - y_{mo} - \ell_{eq}) - kR\sin(\Omega t + \theta_0) = 0$

c) $\ddot{x} + 2\underbrace{\left(\frac{b}{2m}\right)}_{\gamma}\dot{x} + \underbrace{\left(\frac{k}{m}\right)}_{\omega_0^2}x = \underbrace{\left(\frac{kR}{m}\right)}_{B}\sin(\Omega t + \theta_0)$

d) $A_p(\Omega) = \frac{0.48}{\sqrt{3{,}52\Omega^2 + (\Omega^2 - 12.13)^2}}$

Problem 6.4.9. A block of 2 kg mass moves in a horizontal plane in the direction of the x-axis, under the action of a restoring force of recovery constant $18\,\mathrm{N/m}$ and in the presence of a viscous friction force of coefficient $16\,\mathrm{N\,s/m}$.

By means of an engine, a harmonic force $F = F_0 \sin\Omega t$ is applied to the block, also in the x direction. Under stationary conditions, the expression for the trajectory of the block is $x = 0.01\sin(\Omega t - \pi/2)$ (x in m and t in s).

a) Calculate the pulsation Ω, the mechanical impedance and the amplitude of the force.

b) Express the velocity of the block as a function of time and calculate the phase difference between the velocity and the force.

Now, the engine is switched off:

c) What kind of motion does the block make? Give reasons and numerical justification for the answer.

Solution: a) 3 rad/s; 16 kg/s; 0.48 N; b) $0.03 \sin 3t$; 0

Problem 6.4.10. Consider the system in the figure. When in equilibrium, $\ell = \ell_{\text{eq}}$. The pulley rotates about a fixed axis passing through O with moment of inertia I. It is affected by viscous friction with a moment about the shaft $M_{\beta(O)} = -\beta\,\dot{\phi}$, where $\dot{\phi}$ is the angular velocity and β is a constant.

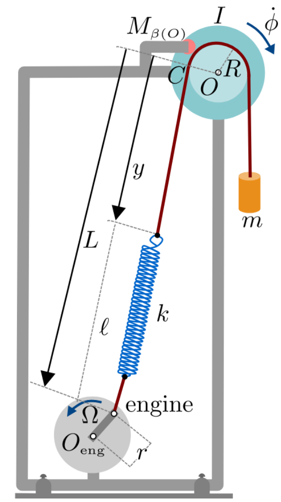


Figure for Problem 6.4.10

We assume that, at all times, the rope remains taut and does not slip. With the engine stopped, $L =$ ct, we move the mass m vertically and release it:

a) Write the equation of motion using the y-coordinate.

We start the engine in such a way that L is no longer constant and can be expressed as $L = \overline{CO}_{\text{eng}} + r\sin(\Omega t + \theta_0)$ due to the fact that $\overline{CO}_{\text{eng}} \gg r$ (C is the rope-pulley contact point, which in this approximation is at rest).

b) Find the new equation of motion.

c) Write the equation of motion in canonical form (call x the new coordinate) and identify all the parameters involved.

d) Plot the amplitude of the forced stationary oscillations as a function of engine pulsation Ω.

Data:

$$R = 14\text{ mm};\ I = 6.0\times10^{-5}\text{ kg m}^2;\ r = 20\text{ mm}$$
$$m = 107\text{ g};\ k = 3.50\text{ N/m};\ \beta = 1.81\times10^{-4}\text{ N m s}$$

Solution (in SI units):

a) $\left(\frac{I}{R^2} + m\right)\ddot{y} + \frac{\beta}{R^2}\dot{y} + k(y + \ell_{\text{eq}} - L) = 0$

b) $\left(\frac{I}{R^2} + m\right)\ddot{y} + \frac{\beta}{R^2}\dot{y} + k(y + \ell_{\text{eq}} - \overline{CO}_{\text{eng}}) - kr\sin(\Omega t + \theta_0) = 0$

c) $$\ddot{x} + 2\underbrace{\left(\frac{\beta}{2R^2\left(\frac{I}{R^2} + m\right)}\right)}_{\gamma}\dot{x} + \underbrace{\left(\frac{k}{\frac{I}{R^2} + m}\right)}_{\omega_0^2}x = \underbrace{\left(\frac{kr}{\frac{I}{R^2} + m}\right)}_{B}\sin(\Omega t + \theta_0);$$

$x = y + \ell_{\text{eq}} - \overline{CO}_{\text{eng}}$

d) $A_p(\Omega) = \frac{0.169}{\sqrt{5{,}00\Omega^2 + (\Omega^2 - 8.47)^2}}$

Problem 6.4.11. In a liquid of density $\rho = 10^3\text{kg/m}^3$, a bottom-ballasted buoy of total mass $m = 20$ kg is held in equilibrium at $y = 0$, where y is a vertical coordinate. To avoid excessive oscillations, it is designed to have a damping of

$-80\ \dot{y}$ (SI units). The cylindrical part of the buoy, of radius $R = 0.25$ m, always touches the water level. Determine:

a) The differential equation of motion for the y-coordinate.

b) The period of the oscillations.

c) The position as a function of time, $y(t)$, if we hit it when it is in equilibrium so that the initial velocity is $\dot{y}_0 = -10$ m/s.

Due to a smooth and persistent swell, it receives a vertical excitation force $F = 100\ \sin(10t)$ (SI units). Determine:

d) The position as a function of time, $y(t)$, for the stationary motion.

Solution: a) $\ddot{y}+4\dot{y}+96.3y = 0$; b) $T = 0.654$ s; c) $y(t) = 1.041\ \mathrm{e}^{-2t}\sin(9.60\ t+\pi)$; d) $y(t) = 0.1244\ \sin(10t - 1.66)$

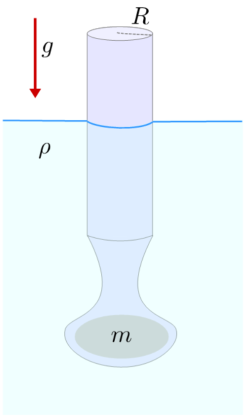


Figure for Problem 6.4.11

Problem 6.4.12. A 2 kg particle attached to a spring of recovery constant of 18 N/m and a damper with damping parameter $\gamma = 3\ \mathrm{s}^{-1}$ moves in a rectilinear motion in the x-direction, where $x = 0$ is its equilibrium point.

a) Write the equation of motion.

At the initial instant, the particle is at position $x = 0.3$ m, with velocity $\dot{x} = -0.2$ m/s.

b) Write the expression of its trajectory

We apply a harmonic force in the x-direction to the particle so that it oscillates with a constant amplitude of 0.1 m and takes 0.25 s to go from one end of the oscillation to the other.

c) What must the value be for the amplitude of this force?

d) What is the maximum speed of the particle under these conditions?

Solution: a) $\ddot{x} + 6\dot{x} + 9x = 0$; b) $x(t) = (0.3 + 0.7t)\,\mathrm{e}^{-3t}$; c) $F_0 = 33.383$ N; d) 1.257 m/s

→ 7P

7 Problems and questions

Problem 7.2.3. The photograph of a wave pulse on a string at the instant $t = 0$ indicates that the shape of the pulse is (SI units)

$$y(x,0) = \frac{18 \times 10^{-3}}{8 + x^2}$$

If the elongation of the string at the point $x = 5.20$ m and at the instant $t = 0.40$ s is $2.25\,\text{mm}$, at what speed does this wave propagate? If the linear density of the string is $0.280\,\text{kg/m}$, to what tension is it subjected?

Solution: $13.0\,\text{m/s}$, $47.3\,\text{N}$

Problem 7.2.4. If A, B, C, k, ω and ϕ are constants, which of the following functions represent waves? What is the propagation speed in the positive cases?

a) $y(x,t) = A\cos^2(kx - \omega t + \phi)$ **b)** $y(x,t) = A\cos kx\, \cos\omega t$
c) $y(x,t) = \frac{A}{(B\,x + C\,t)^2 + 1}$ **d)** $y(x,t) = \frac{A}{(B\,x^2 - C\,t^2) + 1}$

Solution: The functions **a**, **b** and **c** are waves. The velocities are: for **a** and **b**, $v = \frac{\omega}{k}$; for **c**, $v = \frac{C}{B}$.

Problem 7.2.5. Answer the following related questions:

a) Write a harmonic wave propagating towards decreasing x, $8.0\,\text{mm}$ in amplitude, $230\,\text{Hz}$ in frequency and $145\,\text{m/s}$ in speed.

b) What is the distance between two points which, at a given instant, are out of phase by $\pi/3\,\text{rad}$?

c) What is the phase difference of the elongation at the same point between two instants of time separated by 1.5×10^{-3} s?

Solution: a) $y = 8.0 \times 10^{-3} \sin(9.966x + 1445t)$; b) $0.1051\,\text{m}$; c) $2.168\,\text{rad}$

Problem 7.2.6. A harmonic wave passes through two points on a string, x_1 and x_2, which are separated by 1.20 m and vibrate, respectively (in SI units),

$$y_1 = 0.020 \sin \pi \left(3t - \frac{1}{2}\right) \qquad , \qquad y_2 = 0.020 \sin \pi \left(3t - 1\right)$$

Calculate the velocity at which the wave propagates, the wavelength and the wave function.

Solution: $7.20\,\mathrm{m/s}$, $4.80\,\mathrm{m}$; $y(x,t) = 0.020\ \sin 3\pi(t - x/7.20 - 1/6)$

Problem 7.2.7. Consider the wave $y(x,t) = 4\cos\left[2\pi\left(\frac{t}{6} + \frac{x}{240}\right)\right]$, where y and x are expressed in cm and t in s. Calculate:

a) The phase difference, at a given instant, between two particles in the medium separated by 210 cm.

b) The phase difference between two positions and the same instant knowing that the particle in the medium takes 1.0 s to go from one to the other of these positions.

c) If, at a given instant, a given particle has an elongation of 3.0 cm, what will its elongation be 2.0 s later?

Solution: a) $7/4\pi$ rad; b) $\pi/3$ rad; c) -3.79 cm

Problem 7.2.8. A transverse harmonic wave propagates along an indefinite string with a speed of $4.0\,\mathrm{m/s}$. At all times, the minimum distance between two points in phase is 20 cm. It is known that, at the origin $x = 0$ and at the initial instant $t = 0$, the elongation is maximum with a value of 20 cm. Find:

a) The amplitude, wavelength and period.

b) The elongation and velocity of a point $x = 0.25$ m, after $t = 5/16$ s.

c) The minimum distance between two points with a phase difference of $\pi/3$ rad.

d) The phase difference between two points separated by $\Delta x = 5$ cm,

Solution: a) 20 cm; 0.20 m, 0.05 s; b) 20 cm; $0\,\mathrm{m/s}$; c) 3.33 cm; d) $\pi/2$ rad

Problem 7.2.9. A harmonic plane wave travels with a propagation speed of $32\,\mathrm{m/s}$. The amplitude is 2.3 cm and the frequency is 60 Hz. Assuming that at the origin $x = 0$ and at the initial instant $t = 0$ the elongation is maximum, what are the values of the elongation, velocity and acceleration at a point $x = 15.3$ m, after $t = 2.60$ s have elapsed.

Solution: $-0.88\,\text{cm}$; $-8.01\,\text{m/s}$; $1251\,\text{m/s}^2$

Problem 7.3.2. A long string, of linear density $0.10\,\text{kg/m}$ and subjected to $25\,\text{N}$ of tension, is made to vibrate at a frequency of $20\,\text{Hz}$ and causes the propagation of a harmonic wave of amplitude $1\,\text{cm}$.

a) Calculate the speed at which the wave propagates and its wavelength.

b) Write the wave function of this wave, knowing that at the initial instant $t = 0$ the elongation of the string is $0.50\,\text{cm}$ at the origin point $x = 0$.

c) At $t = 4\,\text{s}$, what is the elongation, velocity and transverse acceleration of the point on the string at $x = 90\,\text{cm}$?

Solution: a) $15.81\,\text{m/s}$; $0.7905\,\text{m}$; b) $y = 1\times10^{-2}\sin(\omega t - kx + \pi/6)$, $125.66\,\text{rad/s}$; $7.948\,\text{rad/m}$; c) $-0.3383\,\text{cm}$; $118.3\,\text{cm/s}$; $53.42\,\text{m/s}^2$

Problem 7.3.3. Using dimensional analysis, find the expressions giving the propagation velocities v of the following two types of waves:

a) Waves on a very long string, of linear density μ and subjected to a tension F.

b) Waves on the surface of a lake or sea caused by the weight of the water when the vertical amplitude of the waves is much smaller than the depth h of the water. The quantities on which these surface waves may depend are the water density, ρ, the acceleration of gravity g and the depth, h.

Solution: a) $v = k\sqrt{\frac{F}{\mu}}$; b) $v = k\sqrt{gh}$, where k is a dimensionless constant (a detailed physical study shows it to be 1).

Problem 7.3.4. A long, heavy chain of length L and mass m is hung from the ceiling. One end is struck by a hand, causing a wave pulse that rises upwards, reaches the ceiling, is reflected and returns back to the original end. Calculate how long it will take for the pulse to go up and down.

Solution: $\Delta t = 4\sqrt{\frac{L}{g}}$

Problem 7.3.5. The speed of sound waves in air is given by (7.29):

$$v = \sqrt{\frac{\gamma RT}{M}}$$

where $\gamma = 1.40$, $R = 8.314\,\text{J/molK}$ and M, the molar mass of air, is $0.0290\,\text{kg/mol}$. Making the necessary approximations, find a simple expression for the speed of sound as a function of the temperature t_C of the air in degrees Celsius, °C, for temperatures near 0°C.

Solution: In m/s, $v(t_C) = 331 + 0.606 t_C$, if t_C in °C.

Problem 7.3.6. When a 200 m long steel tube is struck at one end, a person at the other end hears two sounds resulting from two longitudinal waves, one propagating through the tube and the other through the air, at $T = 20$°C. What is the time interval between both sounds?

Solution: 0.549 s

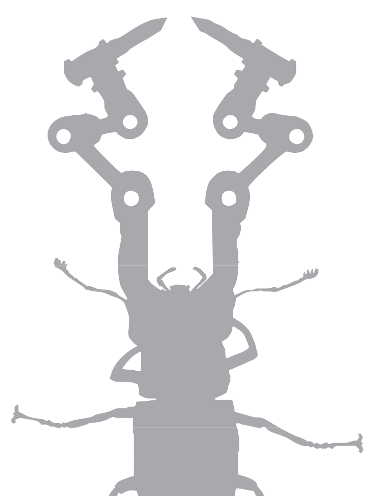
Problem 7.4.1. Using a mathematics programme, graphically check that the following two periodic functions are equivalent to the developments in the indicated Fourier series:

a) *Absolute value of* $\sin(\omega t)$: $f(t) = |\sin \omega t|$

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi} \cdot \left(\frac{\cos 2\omega t}{1 \cdot 3} + \frac{\cos 4\omega t}{3 \cdot 5} + \frac{\cos 6\omega t}{5 \cdot 7} + ... \right)$$

b) *Positive triangular saw*: $f(t) = t,\ 2\pi n < t < 2\pi(n+1)\,, n = 0, 1...$

$$f(t) = \pi - 2\left(\sin t + \frac{1}{2} \sin 2t + \frac{1}{3} \sin 3t + ... \right)$$



→ 8P

8 Problems and questions

Problem 8.1.3. We vibrate a tensioned wire in such a way that we generate in it transverse waves with a frequency of $120\,\mathrm{Hz}$ and an amplitude of $2.30\,\mathrm{mm}$. The density of the wire is $0.010\,\mathrm{kg/m}$ and it is subjected to a tension of $100\,\mathrm{N}$.

a) What are the angular frequency and the wave number?

b) What is the energy per unit length of the wire? What is the minimum power that must be supplied to keep the amplitude of the waves constant?

Solution: a) $754.0\,\mathrm{rad/s}$; $7.540\,\mathrm{rad/m}$; b) $0.01504\,\mathrm{J/m}$; $1.504\,\mathrm{W}$

Problem 8.2.2. Longitudinal waves of amplitude $1.0 \times 10^{-5}\,\mathrm{m}$ and frequency $30\,\mathrm{Hz}$ propagate through a steel bar with a diameter of $4\,\mathrm{cm}$. Calculate:

a) The wave function propagating along the bar.

b) The energy of the rod per unit volume.

c) The average power and intensity propagating through the bar.

Solution: a) $s = 1.0 \times 10^{-5} \sin(0.03722\,x - 188.5t)$ (SI units); b) $0.0139\,\mathrm{J/m^3}$; c) $88.2\,\mathrm{mW}$; $70.2\,\mathrm{W/m^2}$

Problem 8.3.2. Answer the following related questions.

a) What is the intensity and intensity level of a sound wave in air with a sound pressure amplitude of $0.20\,\mathrm{Pa}$? What is this amplitude as a percentage of the atmospheric pressure?

b) To which displacement wave does it correspond if the frequency of the sound is $440\,\mathrm{Hz}$?

Note: Assume that the acoustic impedance of air is 418 rayl.

Solution: a) 4.6×10^{-5} W/m^2; 76.6 dB; 2.0×10^{-4} %; b) 1.73×10^{-7} m

Problem 8.3.3. A plane sound wave in air has a frequency of 100 Hz and an amplitude of 7.7×10^{-6} m. If the air pressure is 1 atm = 1033 hPa and the temperature is 15°C, determine:

a) The acoustic impedance of the air.

b) The intensity and intensity level of the sound wave.

c) The amplitude of the corresponding acoustic pressure wave.

Solution: a) 425 rayl; b) 4.97×10^{-3} W/m^2; 97.0 dB; c) 2.06 Pa

Problem 8.3.4. Knowing that the acoustic impedances of air and water are, respectively, 418 rayl and 1.45×10^6 rayl, calculate:

a) The acoustic pressure amplitude ratio of two waves, one in water and the other in air, that have the same frequency and equal intensities.

b) If the acoustic pressure amplitudes of both waves were equal, what would be the ratio of their intensities?

Solution: a) 58.9; b) 2.88×10^{-4}

Problem 8.3.5. The sound of an explosion lasting 0.25 s is no longer audible 80 km from the point where it occurred. The threshold intensity for this sound is 8×10^{-12} W/m^2. Assuming that the sound propagates as a spherical wave with no energy loss, determine:

a) The acoustic energy involved in the sound of the explosion.

b) The distance at which the intensity level is 50 dB.

c) At the distance found for question b), how many simultaneous explosions would be necessary for the intensity level to be 70 dB?

Solution: a) 0.161 J; b) 253 m; c) 100

Problem 8.3.6. The intensity of a plane wave has been reduced by 30% after passing through 12 cm of absorbing material.

a) What is the absorption coefficient of the material for this type of wave?

b) How much of the material did the wave pass through at the instant when the intensity was 90% of the initial intensity?

Solution: a) $2.97\,\mathrm{m}^{-1}$; b) $3.55\,\mathrm{cm}$

Problem 8.3.7. Two wires of different densities are welded together and subjected to a given tension. A wave propagates through the first wire and, upon reaching the weld, one part is reflected and the other is transmitted. Knowing that the amplitude of the reflected wave is half that of the transmitted wave, and that the velocity of the waves in the first wire is twice that of the second, calculate:

a) The amplitude ratio of the three waves.

b) The percentage of incident power transmitted and the percentage that is reflected.

Solution: a) $a_R = \frac{A_R}{A_I} = \frac{1}{3}$; $a_T = \frac{A_T}{A_I} = \frac{2}{3}$; b) $p_R = \frac{P_R}{P_I} = \frac{1}{9}$; $p_T = \frac{P_T}{P_I} = \frac{8}{9}$

Problem 8.3.8. A longitudinal harmonic wave passes through a steel $\rightarrow$ copper interface. The incident wave has a period of 1.0×10^{-3} s, a wavelength of $5.05\,\mathrm{m}$ and an amplitude of $2.0 \times 10^{-6}\,\mathrm{m}$. The transmitted wave is observed to have a wavelength of $3.71\,\mathrm{m}$. Calculate:

a) The propagation speed of the waves in each medium.

b) The steel $\rightarrow$ copper transmission and reflection coefficients.

c) The proportion of energy that is reflected and that is transmitted.

d) The amplitude of the reflected wave.

Data: The densities of the steel and copper used are, respectively, $\rho_{\text{steel}} = 7850\,\mathrm{kg/m^3}$, $\rho_{\text{Cu}} = 8960\,\mathrm{kg/m^3}$. Young's moduli are given in Table 7.1.

Solution: a) $v_{\text{steel}} = 5050\,\mathrm{m/s}$; $v_{\text{Cu}} = 3710\,\mathrm{m/s}$; b) 1.088; 0.0878; c) 0.771%; 99.23%; d) $1.76 \times 10^{-7}\,\mathrm{m}$

Problem 8.3.9. In a hospital ultrasound system, the transducer (the part that emits the ultrasound) is made of aluminium.

a) Considering that the ultrasound comes out of the aluminium, passes through the air and reaches the body, what percentage of the ultrasound power enters the body?

b) If we now put glycerine instead of air between the aluminium transducer and the body, how much is the percentage of the ultrasound power entering the body?

Data: The acoustic impedance of the body is the same as for water, 1.45×10^{6} rayl;

the impedances for air aluminium and glycerine are, respectively, $418\,\text{rayl}$ and 13.7×10^6 rayl, 2.46×10^6 rayl.

Solution: a) $1.41 \times 10^{-5}\%$; b) 48.2%

Problem 8.3.10. Consider two cables of cross-sections S_1 and S_2 and linear densities μ_1 and μ_2, welded at a point and subjected to a tension F. The expressions for the transmitted and reflected amplitudes and powers of the transverse waves are governed by the same expressions as in the longitudinal case, with the corresponding impedance: $Z_i = \rho_i v_i = \frac{\mu_i}{S_i}\sqrt{\frac{F}{\mu_i}} = \sqrt{F}\frac{\sqrt{\mu_i}}{S_i}$.

Apply this knowledge to the following case:

Two wires, one of copper and the other of steel, each with a radius of 1 mm, are joined together to form a longer cable. The tension of the assembly is $50\,\text{N}$. A $10\,\text{Hz}$ wave propagates from the copper to the steel with an amplitude of $2.0\,\text{mm}$.

a) Calculate the wavelength of the wave in each wire.

b) Calculate the transmission and reflection coefficients.

Data: Density of copper: $\rho_{\text{Cu}} = 8900\,\text{kg/m}^3$; density of steel: $\rho_{\text{Ac}} = 7800\,\text{kg/m}^3$

Solution: a) $4.23\,\text{m}$; $4.52\,\text{m}$; b) 1.033; 0.033

Problem 8.4.3. A harmonic wave of 1 cm amplitude is superimposed on another wave of 2 cm amplitude, out of phase with respect to the first wave by $-\pi/3\,\text{rad}$. What are the amplitude and phase difference of the resulting wave with respect to the first?

Solution: $2.646\,\text{cm}$, $-0.7137\,\text{rad}$

Problem 8.4.4. At the points $S_1 = (0, 3)$ and $S_2 = (4, 0)$ (SI units), there are two coherent sources of $100\,\text{Hz}$ spherical sound waves. The amplitudes at a distance of 1 m from the sources are 1×10^{-3} Pa and 3×10^{-3} Pa, respectively. The sound propagates at $340\,\text{m/s}$.

a) If the sources emit in phase, what is the amplitude of the sound pressure at the origin $(0, 0)$?

b) By how much would the second source have to advance relative to the first one in order for there to be constructive interference at the origin $(0, 0)$?

Solution: a) 0.190×10^{-3} Pa; b) $1.85\,\text{rad}$

Problem 8.4.5. As shown in the figure, the sound of a 440 Hz tuning fork enters a tube at A, bifurcates into two waves that follow the two paths ABD and ACD, which then meet again at point D, where they interfere with each other. The length of the ABD path is 250 cm and, although the initial maximum length of the ACD path is also 250 cm, it slowly decreases to 75 cm while the tuning fork continues to sound.

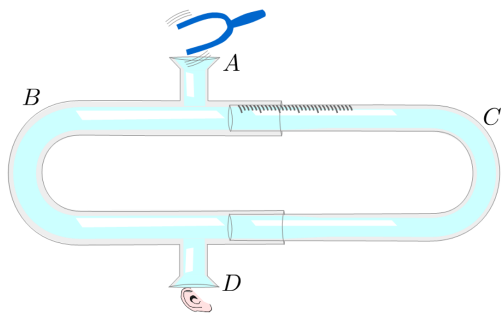


Figure for Problem 8.4.5

If the speed of sound is 340 m/s, through how many minima and maxima does the resulting sound intensity pass during the reduction in the length of ACD?

Solution: Maxima, 2: ACD = 173 cm; 95 cm; minima, 2: ACD = 211 cm; 134 cm.

Problem 8.4.6. Two loudspeakers aligned with a person coherently emit plane sound waves of the same frequency: 440 Hz. The speed of sound is 340 m/s.

a) If they emit in phase, how far apart do the speakers have to be from each other for the person to hear nothing?

b) If they are still emitting coherently but now the phase of the closest speaker is advanced $\pi/3$ with respect to the other one, by how much do the above distances change?

Solution: a) 0.39 m, 1.16 m; 1.93 m...; b) All are reduced by 0.13 cm

Problem 8.4.7. In a laboratory where the speed of sound is 340 m/s, two loudspeakers, A and B, emit plane sound waves of 791 Hz while facing each other at a distance of 11 m, as shown in the figure. A sound level meter C is located on the line between them, 5 m from B. Speaker A provides an intensity of 0.75 W/m^2 and speaker B of 0.25 W/m^2.

a) If the two loudspeakers emit without coherence, what is the value of the intensity level β_0 recorded by sound level meter C?

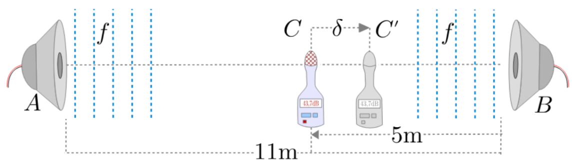


Figure for Problem 8.4.7

b) If the loudspeakers emit coherently and in phase, what will the phase difference be between the sound from A and B when it reaches sound level meter C?

c) What minimum distance δ_1 should we move the sound level meter C towards B in order for the intensity to experience a maximum? And what δ_2 to experience a minimum?

d) What level of intensity will the sound level meter measure at the maximum and at the minimum?

Solution: a) 120 dB; b) 2.051 rad; c) $\delta_1 = 14.5$ cm; $\delta_2 = 3.7$ cm; d) 122.7 dB; 111.3 dB

Problem 8.4.8. Two loudspeakers S_1 and S_2 are located on the y-axis at $y = \pm d/2$, as shown in the figure. Using an audio-frequency amplifier, they are made to emit sounds of frequency f in phase.

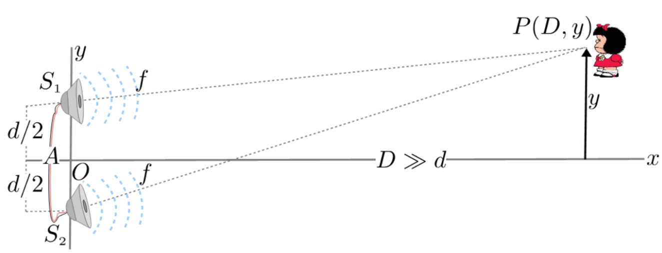


Figure for Problem 8.4.8

An observer moves from $y = 0$ along a line parallel to the y-axis and situated at a great distance D from this axis. If $d \ll D$ and $y \ll D$:

a) Demonstrate that the observer will perceive the first maxima of sound intensity at the distances

$$y_{\text{const}} = n\,\frac{D}{d}\,\frac{v}{f} \qquad \text{with} \qquad n = 0,\ 1,\ 2,\ \ldots$$

where v is the speed of sound.

b) With $d = 2\,\mathrm{m}$ and $D = 80\,\mathrm{m}$, and assuming $v = 340\,\mathrm{m/s}$, what is the frequency for which the distance between two consecutive maxima of intensity is 3.0 m?

Solution: $4.53\,\mathrm{kHz}$

Problem 8.4.9. Two point sources $A = (0, 4, 0)$ and $B = (8, 0, 0)$ (SI units) and of, respectively, 1 mW and 2 mW power, emit spherical waves in phase in open air. If the speed of sound is $340\,\mathrm{m/s}$, find out:

a) The phase difference at which the two waves arrive at the origin if the frequency of both sources is 200 Hz.

b) The frequencies between 200 Hz and 1500 Hz at which the two foci would have to emit simultaneously in order for there to be destructive interference at the origin.

c) The intensities at which the two waves arrive—separately— at the source.

d) How much the intensity of sound is if the foci emit at a frequency that cause destructive interference at the origin. Find the same when there is constructive interference.

Solution: a) 14.784 rad; b) 212.5 Hz; 255.0 Hz ... 1.445 kHz; 1.488 kHz;
c) $4.97\,\mu\mathrm{W/m^2}$; $2.49\,\mu\mathrm{W/m^2}$; d) $0.427\,\mu\mathrm{W/m^2}$; $14.50\,\mu\mathrm{W/m^2}$

Problem 8.5.4. The fundamental frequency of a given violin string of length L is 196 Hz. If the violinist wants to obtain a fundamental frequency of 440 Hz, how long does the string have to be?

Solution: $0.445L$

Problem 8.5.5. The first and last strings on a piano are tuned to 33 Hz and 4186 Hz, with lengths of, respectively, 198 cm and 5.1 cm. If the two strings are under the same tension, how much is the quotient of the effective linear densities of the two strings?

Solution: 10.68

Problem 8.5.6. The figure shows a rod F making sinusoidal vibrations at a frequency of 100 Hz. This excites the horizontal string AB of length 120 cm, tensioned by mass M, which has a weight of 2.25 N, and remains practically immobile. A system of standing waves is thus obtained, with a node in the immediate vicinity of end A on the rod and another node at point B, which is in contact with the pulley. Between these two nodes are four antinodes. The amplitude of the vibrations of the antinodes is 10 mm.

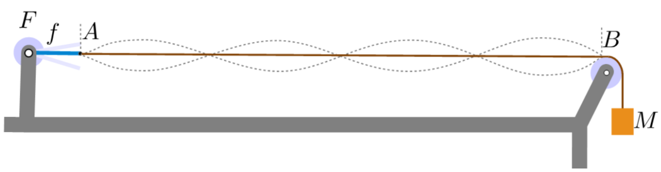


Figure for Problem 8.5.6

Determine:

a) The wavelength of the vibrations and their propagation velocity.

b) The maximum velocity of a point on the string corresponding to an antinode.

c) The amplitude of the vibrations of the point on the string located 35 cm from end A.

d) The weight of a new mass M' hanging from the string if we wanted to obtain three antinodes instead of four.

Solution: a) 0.60 m; 60 m/s; b) 6.28 m/s; c) 5 mm; d) 4 N

Problem 8.5.7. A standing wave on a 6.00 m long elastic string is described by the function (SI units):

$$y(x,t) = 2.0 \times 10^{-2} \sin\left(\frac{\pi}{2}x\right) \cos(\pi t)$$

a) Schematically represent the standing wave, indicating the positions of the nodes and of the bellies. In which harmonic does this wave vibrate?

b) Calculate the propagation speed of the plane wave.

c) Write the harmonic wave functions that generate this standing wave.

d) At what instants will the string be completely straight?

e) At which points on the string and for which instants will the transverse velocity be maximum? What value will this velocity have?

f) When will the transverse velocity of the wave be zero?

g) By what percentage should the tension be increased to form standing waves with one less belly?

Solution: a) 3_d harmonic; b) $v = 2.00$ m/s; c) $y_+(x,t) = 1.0 \times 10^{-2} \sin(x/2 - \pi t)$; $y_-(x,t) = 1.0 \times 10^{-2} \sin(x/2 + \pi t)$; d) $t = n + 1/2$ s; $n = 0,\ 1,\ 2,\ 3...$; e) In the antinodes, $x_v = 1,\ 3,\ 5\,\text{m}$; for $t = n + 1/2\,\text{s}$; $v = 6.28\,\text{m/s}$; f) $t = n\,\text{s}$; g) 125%

Problem 8.5.8. Two identical strings of mass 100 g and length 1 m are fixed at the ends and subjected to tensions of, respectively, 200 N and 205 N.

a) If the strings vibrate in the third harmonic, calculate the frequency of the beats resulting from the superposition of the sounds generated by each string.

Then, only the 200 N tension string is made to vibrate at the fundamental harmonic. If the amplitude at a point 20 cm from one end is 1.0 cm, determine:

b) The maximum velocity of the transverse motion at this point on the string.

c) The elongation at this point at the instant $t = 0$, knowing that at this instant the elongation at the centre of the string is maximum.

Solution: a) 0.833 beats/s; b) $\dot{y} = 1.41\,\text{m/s}$; c) $y = 1.00\,\text{cm}$

Problem 8.5.9. The end A of a horizontal string is fixed to the wall and the other end B passes through a frictionless pulley and is attached to a body of mass M, which hangs from it.

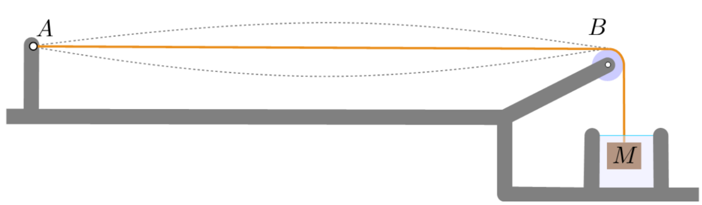


Figure for Problem 8.5.9

The frequency of the fundamental sound emitted by the string is 392 Hz. If the body is completely immersed in water (see Figure), the frequency drops to 343 Hz. Calculate the density of the body.

Solution: $4270\,\text{g/cm}^3$

Problem 8.5.10. A string of 28.28 cm length and 0.050 kg/m linear density is attached to a second string with a linear density that is half that of the first.

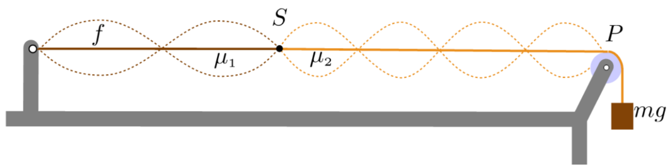


Figure for Problem 8.5.10

One end is fixed to a wall and the other end passes through a pulley and is attached to a weight of 100 N, which hangs from it. Between joint S and the pulley P, the length of this second string is 100 cm. We want standing waves to form along both strings so that there is a node at junction S, as shown in the figure.

What is the lowest frequency we can apply to the strings? In this case, how many antinodes will there be along the two strings?

Solution: 158.1 Hz; 7 antinodes

Problem 8.5.11. A string of 1.40 m length and 2.00 g mass is attached to another string of 1.00 m length and 4.98 g mass. The assembly is made to vibrate at 120 Hz.

a) What is the maximum tension they must be subjected to so that standing waves form when the weld is at a node? How many antinodes will each string have in this case?

b) What other lower tensions will also satisfy this condition?

Solution: a) 17.9 N; 3 and 4 antinodes; b) 4.48 N; 1.99 N...

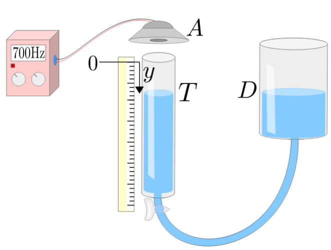


Figure for Problem 8.5.12

Problem 8.5.12. We want to measure the speed of sound in the air. To do this, we use a setup as shown in the figure, where we have a loudspeaker A, that emits harmonic sound that we can control by maintaining the frequency anywhere we want between 400 and 1200 Hz. The speaker faces the mouth of a 1 m long tube T filled with water up to a height that we can modify by raising or lowering the tank D.

The length of the air in the tube, y, is measured with a ruler parallel to the tube.

With the loudspeaker operating at 700 Hz, we lower the water level and observe that, when $y_1 = 20.3$ cm, there is resonance. If we continue lowering the water, the next level at which there is resonance is $y_2 = 44.6$ cm.

a) How much is the speed of sound in the air inside the tube?

b) With a frequency of 1000 Hz and with the water completely filling the tube, we lower the level until we reach the first resonance position and $y_1 = 10.0$ cm. Find all the other positions at which the tube will resonate.

c) If the tube is then left with a fixed column of air, $y = 50$ cm, at what frequencies will the tube resonate?

Solution: a) $v = 340.2$ m/s; b) $y_2 = 27.0$ cm; $y_3 = 44.0$ cm; $y_4 = 61.0$ cm; $y_5 = 78.0$ cm; $y_6 = 95.0$ cm; c) 510 Hz, 851 Hz; 1191 Hz

Problem 8.5.13. We have a cylindrical tube open at both ends that we excite with sound waves of varying frequency. Two consecutive resonance frequencies are observed: 360 Hz and 540 Hz.

a) If we close one of the ends and excite it again, what are the first three frequencies at which it will now resonate?

b) If the tube is 0.950 m long, what is the temperature of the air inside?

Solution: a) 90 Hz; 270 Hz; 450 Hz; b) 18.2° C

Problem 8.6.2. On a day when the temperature is 35°C, the driver of an express train, travelling at 110 km/h, sees a commuter train travelling on the same track further ahead. To determine the speed at which the commuter train is travelling, the driver sounds his 1000 Hz whistle and listens to the 1060 Hz frequency echo.

a) At what speed does the sound propagate?

b) Assuming there is no wind, at what speed does the commuter train travel?

Solution: a) 352 m/s; b) 73.3 km/h

Problem 8.6.3. On a windless day when the speed of sound is 340 m/s, a siren emitting a sound of 1000 Hz moves away from an observer at rest and towards a cliff at a speed of 36.0 km/h.

What is the difference between the two sound frequencies —direct and reflected from the cliff— that will reach the observer? How fast should the siren be travelling so that 8.0 Hz pulses can be heard?

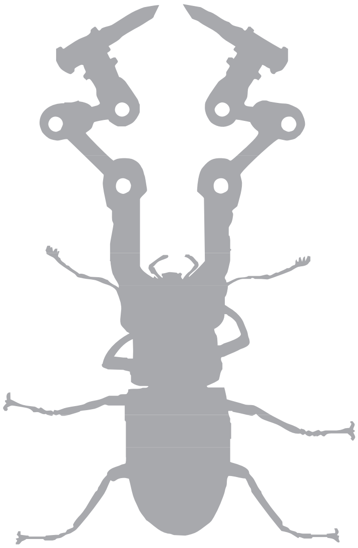
Solution: $58.9\,\text{Hz}$; $4.90\,\text{km/h}$

Problem 8.6.4. An object motion detector could consist of a source of harmonic waves of frequency f_0 and a detector of the frequency of the beats f_b obtained by superimposing the direct waves of the source with the waves reflected by the object. If the object does not move with respect to the source, $f_b = 0$. If it approaches the source or moves away at a velocity v that is much smaller than the velocity c of the waves in the medium, show that the beat frequency is $f_b = 2f_0\frac{v}{c}$.

Problem 8.6.5. A supersonic aircraft travelling horizontally at Mach $= 1.50$ [1] passes through the vertical of an observer in open air on a day when the average speed of sound is $335\,\text{m/s}$. If this observer has to wait $3.20\,\text{s}$ to hear the plane from the time he sees it pass overhead, at what height is the plane flying?

[1] Mach $= M$ is the ratio of the velocity of an object, v, to the velocity of the waves in the medium, c: $M = \frac{v}{c}$

Solution: $1438\,\text{m}$



Solutions to the questions

Chapter 1

1.7.1: **c**, 1.7.2: **e**.

Chapter 2

2.1.1: **b**, 2.1.2: **a**, 2.1.3: **e**.

2.2.1: **d**, 2.2.2: **c**, 2.2.3: **e**, 2.2.4: **c**.

2.3.1: **e**.

2.4.1: **e**, 2.4.2: **a**, 2.4.3: **a**, 2.4.4: **e**, 2.4.5: **b**, 2.4.6: **b**, 2.4.7: **d**, 2.4.8: **c**, 2.4.9: **c**.

2.5.1: **d**, 2.5.2: **b**, 2.5.3: **a**, 2.5.4: **d**, 2.5.5: **c**.

Chapter 3

3.2.1: **c**, 3.2.2: **b**, 3.2.3: **a**, 3.2.4: **c**, 3.2.5: **a**, 3.2.6: **b**, 3.2.7: **c**, 3.2.8: **e**, 3.2.9: **e**, 3.2.10: **c**.

3.3.1: **d**.

3.4.1: **b**, 3.4.2: **d**, 3.4.3: **c**.

Chapter 4

4.3.1: **c**, 4.3.2: **b**, 4.3.3: **c**, 4.3.4: **a**, 4.3.5: **a**, 4.3.6: **d**.

4.5.1: **a**, 4.5.2: **a**, 4.5.3: **c**, 4.5.4: **c**, 4.5.5: **a**, 4.5.6: **c**, 4.5.7: **e**, 4.5.8: **a**, 4.5.9: **e**, 4.5.10: **b**, 4.5.11: **a**, 4.5.12: **c**, 4.5.13: **c**, 4.5.14: **b**, 4.5.15: **e**, 4.5.16: **b**, 4.5.17: **e**.

4.7.1: **b**, 4.7.2: **e**, 4.7.3: **b**.

Chapter 5

5.2.1: **e**, 5.2.2: **d**, 5.2.3: **c**, 5.2.4: **a**. 5.2.5: **c**, 5.2.6: **c**, 5.2.7: **d**, 5.2.8: **c**, 5.2.9: **c**, 5.2.10: **c**, 5.2.11: **b**, 5.2.12: **e**, 5.2.13: **e**, 5.2.14: **e**, 5.2.15: **b**, 5.2.16: **e**.

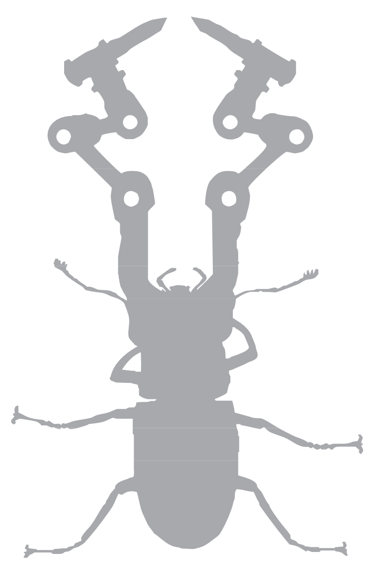
5.3.1: **a**, 5.3.2: **d**, 5.3.3: **a**, 5.3.4: **a**, 5.3.5: **d**, 5.3.6: **d**.

Chapter 6

6.2.1: **b**, 6.2.2: **c**, 6.2.3: **d**, 6.2.4: **d**, 6.2.5: **a**.

6.3.1: **a**, 6.3.2: **e**, 6.3.3: **c**.

6.4.1: **c**, 6.4.2: **b**, 6.4.3: **d**, 6.4.4: **b**, 6.4.5: **d**, 6.4.6: **c**, 6.4.7: **a**, 6.4.8: **b**, 6.4.9: **d**.



Tables

Trigonometric relations

$\sin^2\alpha + \cos^2\alpha = 1$
$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$
$\sin(\alpha \pm \beta) = \sin\alpha\ \cos\beta \pm \cos\alpha\ \sin\beta$
$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\ \tan\beta}$
$\cos(\alpha \pm \beta) = \cos\alpha\ \cos\beta \mp \sin\alpha\ \sin\beta$
$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$
$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$
$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$
$\sin(0) = 0, \sin(\frac{\pi}{6}) = \frac{1}{2}, \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}, \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}, \sin(\frac{\pi}{2}) = 1, \sin(\pi) = 0,$
$\cos(0) = 1, \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}, \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}, \cos(\frac{\pi}{3}) = \frac{1}{2}, \cos(\frac{\pi}{2}) = 0, \cos(\pi) = -1$

Derivatives

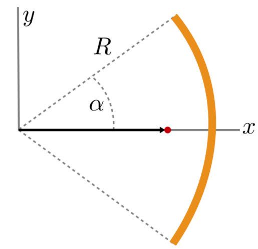
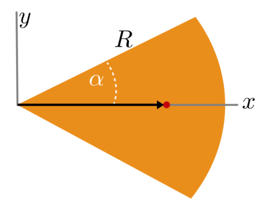
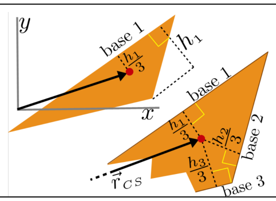
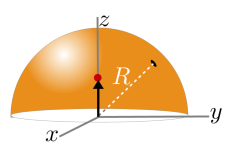
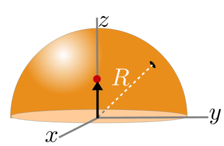
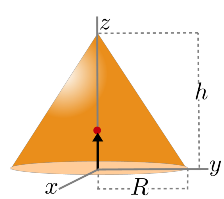
Function: $F(x)$	**Derivative:** $\frac{dF(x)}{dx}$
$a\,f(x) + g(x)$	$a\,\frac{df(x)}{dx} + \frac{dg(x)}{dx}$
$f(x)g(x)$	$\frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx}$
$\frac{f(x)}{g(x)} = f(x)g(x)^{-1}$	$\frac{\frac{df(x)}{dx}g(x) - f(x)\frac{dg(x)}{dx}}{g(x)^2}$
$f(x)^n$	$n\,f(x)^{n-1}\frac{df(x)}{dx}$
$f(g(x))$	$\frac{df(g)}{dg}(x)\ \frac{dg(x)}{dx}$
$f^{-1}(x)$ $f^{-1}(x)$ is the inverse function of f; e.g. $y =$ arcsin x is the inverse of $x = \sin y$	$\left.\frac{1}{\frac{df(y)}{dy}}\right\rvert_{y=f^{-1}(x)}$
e^x	e^x
a^x	$a^x\ \ln a$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\frac{1}{\cos^2 x}$
$\ln x$	$\frac{1}{x}$
$\log_{(a)} x$	$\frac{1}{x \ln a}$

Integrals

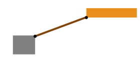
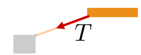
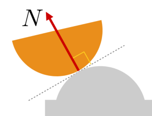
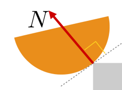
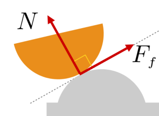
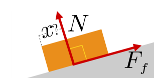
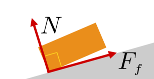
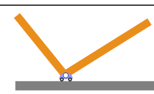
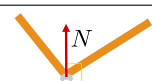
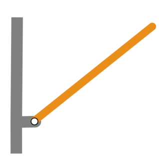
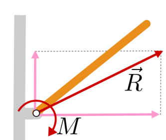
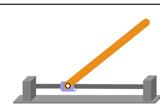
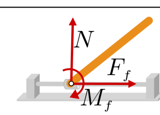
Function: $F(x)$	**Integral:** $\int F(x)\,dx$
$a\,f(x) + g(x)$	$a\,\int f(x)\,dx + \int g(x)\,dx$
x^n with $n \neq -1$	$\frac{1}{n+1}x^{n+1}$
$\frac{1}{x} = x^{-1}$	$\ln x$
e^x	e^x
a^x	$\frac{a^x}{\ln a}$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$-\ln(\cos x)$
$\ln x$	$-x + x\ln x$
$\frac{1}{(x-a)(x-b)}$	$\frac{1}{a-b}\ln\frac{x-a}{x-b}$
$\frac{1}{\sqrt{a^2-x^2}}$	$\arcsin\frac{x}{a}$
$\frac{1}{\sqrt{x^2-a^2}}$	$\ln(x + \sqrt{x^2 - a^2})$

Centre of mass

$\vec{r}_{CM}$ of simple homogeneous bodies with respect to the given reference frames

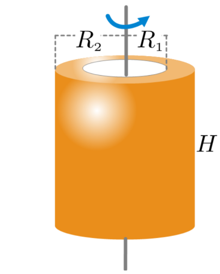
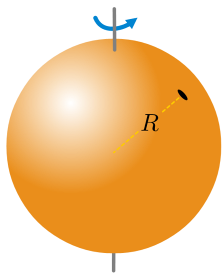
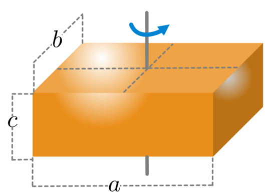
<table>
<tr><td>Arc of circumference</td><td>
</td><td>$$L = 2R\alpha$$ $$\vec{r}_{CS} = \left(\frac{R\sin\alpha}{\alpha}\ ,\ 0\right)$$</td></tr>
<tr><td>Circle sector</td><td>
</td><td>$$S = R^2\alpha$$ $$\vec{r}_{CS} = \left(\frac{2R\sin\alpha}{3\alpha}\ ,\ 0\right)$$</td></tr>
<tr><td>Arbitrary triangle</td><td>
</td><td>$S = \frac{1}{2}b\,h$
The distance from a base i to the CS is $\frac{1}{3}$ of the corresponding height h_i</td></tr>
<tr><td>Semi-sphere (without the bottom cap)</td><td>
</td><td>$$S = 2\pi R^2$$ $$\vec{r}_{CS} = \left(0, 0, \frac{R}{2}\right)$$</td></tr>
<tr><td>Semi-sphere (solid)</td><td>
</td><td>$$V = \frac{2\pi R^3}{3}$$ $$\vec{r}_{CS} = \left(0, 0, \frac{3R}{8}\right)$$</td></tr>
<tr><td>Cone (solid)</td><td>
</td><td>$$V = \frac{\pi R^2 h}{3}$$ $$\vec{r}_{CS} = \left(0, 0, \frac{h}{4}\right)$$</td></tr>
</table>

Support reactions

Cable or tensioned rope. The force generated by a cable is in the direction of the cable. The direction is always on the side of the cable or the force is cancelled out (cable not taut).		
Smooth regular contact. Reaction N normal to the tangent plane to the regular surface at the point of contact.		
Regular/singular smooth contact. Reaction N normal to the tangent plane to the regular surface at the point of contact.		
Rough contacts. A frictional force F_f must be added to the normal reaction N. When the bodies do not move, $\|F_f\| \leq \mu \|N\|$, where μ is the coefficient of friction. The maximum value of $\|F_f\|$ is reached when movement is imminent.		
Extensive contacts. N-reaction normal to the contact surface. It does not have to pass through the centre of mass.		
Extensive contacts (imminent overturning). N-reaction normal to the support surface applied at the point where contact is concentrated.		
Rollers (frictionless). This is the same case as smooth contacts.		
Joint. $\vec{R}$-reaction, generally unknown, which in 2D means two components. If there is friction (between the surfaces in contact in the area of the joint) or no joint but instead jamming, a torque of M moment must be added. Its maximum value will depend on the nature of the contact.		
Guides. Normal reaction to a guide. If the contact is rough, friction F_f must be added and perhaps also a friction torque M_f.		

Moments of inertia

I of simple homogeneous bodies with respect to the indicated axes.

<table>
<tr><td>Cylinder</td><td>
</td><td>thick:
$I = \frac{1}{2} m \left(R_1^2 + R_2^2\right)$
disc: $H = 0$
solid: $R_1 = 0$
thin: $R_1 = R_2$</td></tr>
<tr><td>Sphere</td><td>
</td><td>solid: $I = \frac{2}{5} mR^2$
empty: $I = \frac{2}{3} mR^2$</td></tr>
<tr><td>Orthohedron</td><td>
</td><td>orthohedron / rectangle:
$I = \frac{1}{12} m \left(a^2 + b^2\right)$
bar: $a = 0$ i $c = 0$</td></tr>
</table>

Constants

Name of the constant:	**Value:**
$\pi-$number	3.1415926536
$e-$ number, $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$	2.718281285
Coulomb constant, $k = \frac{1}{4\pi\varepsilon_0}$	$8.987551788 \times 10^9 \ \frac{\mathrm{Nm^2}}{\mathrm{C}}$
Elementary charge , e	$1.602177 \times 10^{-19}\mathrm{C}$
Avogadro's number, N_A	$6.022137 \times \ 10^{23}$
Boltzmann constant, k	$1.380658 \ \times 10^{23} \ \frac{\mathrm{J}}{\mathrm{K}}$
Ideal gas constant, $R = N_A \, k$	$8.31451 \ \frac{\mathrm{J}}{\mathrm{mol\,K}}$
Gravitational constant, G	$6.6726 \times 10^{-11} \frac{\mathrm{N\,m^2}}{\mathrm{kg^2}}$
Electron mass, m_e	$9,109390 \times 10^{-31}\,\mathrm{kg}$
Proton mass, m_p	$1.672622 \times 10^{-27}\,\mathrm{kg}$
Neutron mass, m_n	$1.674929 \times 10^{-27}\,\mathrm{kg}$
Speed of light, c	$2.99792458 \times 10^8 \ \frac{\mathrm{m}}{\mathrm{s}}$
Gravity acceleration at the Earth's surface, g	Standard value: $9.81 \ \frac{\mathrm{m}}{\mathrm{s^2}}$. In Barcelona: $9.804 \ \frac{\mathrm{m}}{\mathrm{s^2}}$
Earth radius, R_T	6370 km
Earth mass, M_T	$5.98 \times 10^{24}\,\mathrm{kg}$